A general class of tobit models

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Resumo. *Uma suposição comum em relação ao modelo tobit padrão é a normalidade da distribuição dos erros. Entretanto, assimetria e bimodalidade podem estar presentes e modelos tobit alternativos devem ser usados. Neste trabalho, propomos um modelo tobit com base na classe de distribuições log-simétricas, que inclui como casos especiais distribuições de caudas mais pesadas/leves e distribuições bimodais. Implementamos uma abordagem baseada no método da máxima verossimilhança para estimatição dos parâmetros e derivamos um tipo de resíduo. Em seguida, discutimos o problema de realizar teste de hipóteses na classe proposta, usando os testes da razão de verossimilhanças e gradiente, que são particularmente convenientes para os modelos tobit, pois não exigem a matriz de informação. Um estudo completo de Monte Carlo é apresentado para avaliar o desempenho dos estimadores de máxima verossimilhança e os testes considerados. Finalmente, ilustramos a metodologia proposta usando um conjunto de dados reais.*

Palavras-chave: Distribuições log-simétricas; Modelo Tobit; Teste da razão de verossimilhanças; Teste gradiente.

JEL: C12; C13; C34.

Abstract. *A common assumption regarding the standard tobit model is the normality of the error distribution. However, asymmetry and bimodality may be present and alternative tobit models must be used. In this paper, we propose a tobit model based on the class of log-symmetric distributions, which includes as special cases heavy/light tailed distributions and bimodal distributions. We implement a likelihood-based approach for parameter estimation and derive a type of residual. We then discuss the problem of performing testing inference in the proposed class by using the likelihood ratio and gradient statistics, which are particularly convenient for tobit models, as they do not require the information matrix. A thorough Monte Carlo study is presented to evaluate the performance of the maximum likelihood estimators and the likelihood ratio and gradient tests. Finally, we illustrate the proposed methodology by using a real-world data set.*

Keywords: Log-symmetric distributions; Tobit model; Likelihood ratio test; Gradient test.

JEL: C12; C13; C34.

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1 Introduction

After its introduction by Tobin (1958), the tobit model has been used extensively in several areas including economics, environmental sciences, engineering, biology, medicine and sociology; see, for example, Barros et al. (2008), Leiva et al. (2007), Villegas et al. (2011), Amemiya (1984), Thorarinsdottir and Gneiting (2010), Helsel (2011) and Martínez-Flores et al. (2013a,b). The tobit model is used to described censored responses and had its motivation based on a study to analyze the relationship between household expenditure on a durable good and household incomes. In this study, Tobin (1958) faced the existence of many cases where the expenditure was zero, which violated the linearity assumption of common regression approaches. Tobin (1958) introduced a regression model whose response was censored at a prefixed limiting value; see Amemiya (1984).

A strong assumption of the tobit model is that the error term is normality distributed, but it is not always the case in many applications; see, for example, Barros et al. (2010, 2018). The normality assumption may not be appropriate to describe the behavior of strictly positive data, as well as bimodal and/or light- and heavy-tailed data. The use of flexible distributions is very important as often real-world data are better modeled by non-normal distributions, especially in the aspect related to the robustness of the results. In the context of censored responses, some authors have emphasized the importance to use more flexible distributions; see, for example, Arellano et al. (2012), Martínez-Flores et al. (2013a,b), Garay et al. (2015), Massuia et al. (2015) and Barros et al. (2010, 2018).

The log-symmetric distribution class was investigated by Jones (2008) and arises when a random variable (RV) has the same distribution as its reciprocal or when the distribution of the logged RV is symmetrical. This class is very useful for modeling strictly positive, asymmetric, bimodal and light- and heavy-tailed data. The class of log-symmetric distributions is a generalization of the log-normal distribution, which provides more flexible alternatives; see, for example, Vanegas and Paula (2016b). Vanegas and Paula (2015) proposed a semiparametric regression model allowing both median and skewness to be modeled, Vanegas and Paula (2016b) discussed some statistical properties of the log-symmetric class of distributions, Vanegas and Paula (2016a) proposed an extension of the log-symmetric regression models used by Vanegas and Paula (2015) considering an arbitrary number of non-parametric additive components to describe the median and skewness and Medeiros and Ferrari (2017) considered the issue of testing hypothesis in symmetric and log-symmetric linear regression models.

A prominent and recent procedure for hypothesis testing in parametric models is the gradient (GR) test, which was proposed by Terrell (2002). This procedure is simple to compute and only involves the score vector and the maximum likelihood (ML) estimates of the parameter vector under the unrestricted and restricted models. Similarly to the generalized likelihood ratio (LR) statistic (Wilks, 1938), the GR statistic is also attractive for censored samples, as is the case of tobit models, since no computation of the information matrix (neither observed nor expected) is required; see, for example, Lemonte and Ferrari (2011).

In this context, the primary objective of this paper is to propose a class of tobit models based on the log-symmetric distribution. The secondary objectives are: (i) to obtain the ML estimators of the model parameters; (ii) to deal with the issue of performing hypothesis testing concerning the parameters of the proposed tobit-log-symmetric model. The LR and GR tests are used for the hypothesis testing purpose; (iii) to carry out Monte Carlo (MC) simulations to evaluate the performance of the ML estimators and the LR and GR tests; and (iv) to conduct a real world data application of the proposed methodology. The proposed tobit-log-symmetric model is a generalization of the tobit-Birnbaum-

Saunders model introduced by Desousa et al. (2018).

The rest of the paper proceeds as follows. In Section 2, we describe briefly the class of log-symmetric distributions and some properties. In Section 3, we formulate the tobit model based on the log-symmetric class, provide estimation, inference and residual analysis based on the ML method. In Section 4, we carry out the mentioned MC simulations and an empirical application with real-world data is done in Section 5. Finally, in Section 6, we discuss conclusions and future research on the topic of this work.

2 Log-symmetric distributions

Consider a continuous and symmetric RV Y having a symmetric distribution with location parameter $\mu \in \mathbb{R}$, dispersion parameter $\phi > 0$, density generator $g(\cdot)$ and probability density function (PDF)

$$f_Y(y; \mu, \phi, g) = \frac{1}{\phi} g\left(\frac{(y-\mu)^2}{\phi^2}\right), \quad y \in \mathbb{R},$$

with $g(u) > 0$ for $u > 0$ and $\int_0^\infty u^{-1/2} g(u) \partial u = 1$; see Fang et al. (1990). In this case, the notation $Y \sim \mathrm{S}(\mu, \phi^2, g)$ is used. The class of log-symmetric distributions arises when we set $T = \exp(Y)$, that is, we obtain a continuous and positive RV T such that the distribution of its logarithm belongs to the symmetric family. The PDF of T is written as

$$f_T(t; \eta, \phi, g) = \frac{1}{\phi t} g(\widetilde{t}^2), \quad t > 0,$$

where $\widetilde{t} = \log\left([t/\eta]^{1/\phi}\right)$ and $\eta = \exp(\mu) > 0$ is a scale parameter. We write $T \sim \mathrm{LS}(\eta, \phi^2, g)$. The density generator g may be associated with an extra parameter ξ (or an extra parameter vector $\boldsymbol{\xi}$). The cumulative distribution function (CDF) of T is given by

$$F_T(t; \eta, \phi, g) = F_Z(\widetilde{t}; \mu, \phi, g),$$

where $F_Z(\cdot)$ is CDF of $Z = (Y - \mu)/\phi \sim \mathrm{S}(0, 1, g)$. Note that the density generator g leads to different log-symmetric distributions. Some members of log-symmetric distributions are the log-normal (Crow and Shimizu, 1988; Johnson et al., 1994), log-logistic (Marshall and Olkin, 2007), log-Laplace (Johnson et al., 1995), log-Cauchy (Marshall and Olkin, 2007), log-power-exponential (Vanegas and Paula, 2016b), log-Student-t (Vanegas and Paula, 2016b), log-power-exponential (Vanegas and Paula, 2016b), log-slash (Vanegas and Paula, 2016b), harmonic law (Podlaski, 2008), Birnbaum-Saunders (Birnbaum and Saunders, 1969; Rieck and Nedelman, 1991), generalized Birnbaum-Saunders (Díaz-García and Leiva, 2005), and F (Johnson et al., 1995) distributions; see Table 1.

Let $T \sim \mathrm{LS}(\eta, \phi^2, g)$, then we have the properties: (P1) $cT \sim \mathrm{LS}(c\eta, \phi^2, g)$, with $c > 0$; (P2) $T^c \sim \mathrm{LS}(\eta^c, c^2\phi^2, g)$, with $c \neq 0$; and (P3) the median of the distribution of T is η. The properties (P1) and (P2) say that the log-symmetric distribution holds the proportionality and reciprocation properties, respectively. Moreover, (P2) is useful to propose modified moment estimators; see Ng et al. (2003) for the Birnbaum-Saunders case. Finally, (P3) can be used to specify a dynamic point process model in terms of the conditional median; see Saulo et al. (2017).

Tabela 1: Density generator $g(u)$ for some log-symmetric distributions.

Distribution	$g(u)$
Log-normal(η, ϕ)	$\propto \exp\left(-\frac{1}{2}u\right)$
Log-Student-t(η, ϕ, ξ)	$\propto \left(1+\frac{u}{\xi}\right)^{-\frac{\xi+1}{2}}, \xi > 0$
Log-power-exponential(η, ϕ, ξ)	$\propto \exp\left(-\frac{1}{2}u^{\frac{1}{1+\xi}}\right), -1 < \xi \leq 1$
Birnbaum-Saunders($\eta, \phi = 4, \xi$)	$\propto \cosh(u^{1/2})\exp\left(-\frac{2}{\xi^2}\sinh^2(u^{1/2})\right), \xi > 0$
Birnbaum-Saunders-t($\eta, \phi = 4, \boldsymbol{\xi} = (\xi_1, \xi_2)^\top$)	$\propto \cosh(u^{1/2})\left(\xi_2\xi_1^2 + 4\sinh^2(u^{1/2})\right)^{-\frac{\xi_2+1}{2}}, \xi_1, \xi_2 > 0$

3 The tobit-log-symmetric model

Consider a censored response variable to the left Y_i for the case i, which is observable for values greater than γ and censored for values smaller than or equal to γ. Then, in the tobit formulation

$$Y_i = \begin{cases} \gamma, & Y_i^* \leq \gamma, \quad i = 1, \dots, m; \\ \boldsymbol{x}_i^\top \boldsymbol{\beta} + \varepsilon_i, & Y_i^* > \gamma, \quad i = m+1, \dots, n, \end{cases} \tag{1}$$

where $Y_i^* = \boldsymbol{x}_i^\top \boldsymbol{\beta} + \varepsilon_i$, m is the number of cases censored to the left, n is the total number of cases, $\boldsymbol{x}_i = (x_{i1}, \dots, x_{ip})^\top$ is an $n \times 1$ vector of covariates fixed and known, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)^\top$ is a $p \times 1$ vector of regression coefficients, and $\{\varepsilon_i\}$ are independent identically distributed (IID) RVs. The tobit-normal (tobit-NO) model is obtained from (1) when ε_i follows a normal distribution with mean zero and variance ς^2, that is, $\varepsilon_i \overset{\text{IID}}{\sim} \mathrm{N}(0, \varsigma^2)$.

Consider the log-symmetric regression model (Vanegas and Paula, 2015)

$$T_i = \eta_i \, \epsilon_i^{\phi_i}, \quad i = 1, \dots, n, \tag{2}$$

where η_i and ϕ_i are median and skewness of the T_i distribution, respectively, and $\{\epsilon_i\}$ are standard log-symmetric distributed IID RVs denoted by $\epsilon_i \overset{\text{IID}}{\sim} \mathrm{LS}(1, 1, g)$. Then, $T_i \overset{\text{IND}}{\sim} \mathrm{LS}(\eta_i, \phi_i^2, g)$. The structures for η_i and ϕ_i are written as

$$\begin{aligned} \eta_i &= \exp(\boldsymbol{x}_i^\top \boldsymbol{\beta}), \quad i = 1, \dots, n, \\ \log(\phi_i) &= \boldsymbol{w}_i^\top \boldsymbol{\zeta}, \quad i = 1, \dots, n, \end{aligned}$$

where $\boldsymbol{x}_i$ and $\boldsymbol{\beta}$ are as in (1), $\boldsymbol{w}_i = (w_{ik}, \dots, w_{ik})$ is an $n \times 1$ vector of covariates for ϕ_i and $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_k)^\top$ is a $p \times 1$ parameter vector. For simplicity reasons, hereafter it is assumed that $\phi_i = \phi$, for $i = 1, \dots, n$.

By applying logarithm in Equation (2), we obtain

$$\underbrace{\log(T_i)}_{Y_i} = \underbrace{\log(\eta_i)}_{\mu_i} + \phi \underbrace{\log(\epsilon_i)}_{\varepsilon_i}, \quad i = 1, \dots, n, \tag{3}$$

where ε_i is standard symmetric distributed, $\varepsilon_i \overset{\text{IID}}{\sim} \mathrm{S}(0, 1, g)$, and $Y_i \overset{\text{IND}}{\sim} \mathrm{S}(\mu_i, \phi^2, g)$. Then, based on

Equations (1) and (3), we propose a tobit model based on the log-symmetric distribution, denoted by tobit-LS, as

$$Y_i = \begin{cases} \gamma, & Y_i^* \leq \gamma, \quad i = 1, \ldots, m; \\ \boldsymbol{x}_i^\top \boldsymbol{\beta} + \varepsilon_i, & Y_i^* > \gamma, \quad i = m+1, \ldots, n; \end{cases} \tag{4}$$

where $Y_i^* = \log(T_i^*) = \boldsymbol{x}_i^\top \boldsymbol{\beta} + \varepsilon_i$, $\boldsymbol{\beta}$ and $\boldsymbol{x}_i$ are as in (1), and ε_i is as in (3).

Consider a sample of size n, $\boldsymbol{Y} = (Y_1, \ldots, Y_m, Y_{m+1}, \ldots, Y_n)^\top$ say, from a tobit-LS model that contains m left-censored data, that is, the values of Y less than a threshold point γ, and $n - m$ complete or uncensored data, namely, values of Y greater than γ. Then, the corresponding likelihood function for $\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \phi)^\top$

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{m} F_Y(\zeta_i^{\mathrm{c}}; \mu_i, \phi, g) \prod_{i=m+1}^{n} \frac{1}{\phi} g\left(\zeta_i^2\right),$$

where F_Y is the CDF of the symmetric distribution and

$$\zeta_i^{\mathrm{c}} = \left(\frac{\gamma - \boldsymbol{x}_i^\top \boldsymbol{\beta}}{\phi}\right) \quad \text{and} \quad \zeta_i = \left(\frac{y_i - \boldsymbol{x}_i^\top \boldsymbol{\beta}}{\phi}\right). \tag{5}$$

By taking the logarithm of (5), we obtain the log-likelihood function for $\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \phi)^\top$, which is given by

$$\ell(\boldsymbol{\theta}) = \sum_i \ell_i(\boldsymbol{\theta}), \tag{6}$$

where

$$\ell_i(\boldsymbol{\theta}) = \begin{cases} \log(F_Y(\zeta_i^{\mathrm{c}}; \mu_i, \phi, g)), & i = 1, \ldots, m; \\ -\log(\phi) + \log(g\left(\zeta_i^2\right)), & i = m+1, \ldots, n. \end{cases}$$

The score vector for $\boldsymbol{\beta}$ and ϕ is given by

$$\dot{\boldsymbol{\ell}}(\boldsymbol{\theta}) = \frac{\partial \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \sum_{i=1}^{n} \dot{\boldsymbol{\ell}}_i(\boldsymbol{\theta}), \quad \text{where } \dot{\boldsymbol{\ell}}_i(\boldsymbol{\theta}) = (\dot{\boldsymbol{\ell}}_{i\boldsymbol{\beta}}^\top(\boldsymbol{\theta}), \dot{\ell}_{i\phi}(\boldsymbol{\theta}))^\top \tag{7}$$

with

$$\dot{\boldsymbol{\ell}}_{i\boldsymbol{\beta}}(\boldsymbol{\theta}) = \begin{cases} -\frac{1}{\phi} \Omega_i \boldsymbol{x}_i, & i = 1, \ldots, m; \\ -\frac{2}{\phi} W_i \zeta_i \boldsymbol{x}_i, & i = m+1, \ldots, n; \end{cases}$$

$$\dot{\ell}_{i\phi}(\boldsymbol{\theta}) = \begin{cases} -\frac{1}{\phi} \Omega_i \zeta_i^{\mathrm{c}}, & i = 1, \ldots, m; \\ -\frac{1}{\phi} - \frac{2}{\phi} W_i \zeta_i^2, & i = m+1, \ldots, n; \end{cases}$$

with $\Omega_i = \frac{\mathrm{d}F_Y(u)/\mathrm{d}u|_{u=\zeta_i^{\mathrm{c}}}}{F_Y(\zeta_i^{\mathrm{c}})}$ and $W_i = \frac{\mathrm{d}g(u)/\mathrm{d}u|_{u=\zeta_i^2}}{g(\zeta_i^2)}$. To obtain the ML estimate of $\boldsymbol{\theta}$ it is necessary to maximize the expression defined in (6) by equating the score vector $\dot{\boldsymbol{\ell}}(\boldsymbol{\theta})$ to zero, providing the likelihood equations. They are solved using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) quasi-

Newton method; see Mittelhammer et al. (2000, p. 199). The corresponding standard errors (SEs) can be approximated by computing the square roots of the diagonal elements of the inverse of the observed Fisher information matrix (Efron and Hinkley, 1978), which is obtained as $\mathcal{J}(\boldsymbol{\theta}) = -\ddot{\boldsymbol{\ell}}(\boldsymbol{\theta})$, where $\ddot{\boldsymbol{\ell}}(\boldsymbol{\theta})$ denotes the Hessian matrix, that is,

$$\ddot{\boldsymbol{\ell}}(\boldsymbol{\theta}) = \frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top} = \sum_{i=1}^{n} \ddot{\boldsymbol{\ell}}_i(\boldsymbol{\theta}), \quad \text{where } \ddot{\boldsymbol{\ell}}_i(\boldsymbol{\theta}) = \begin{bmatrix} \ddot{\ell}_{i\boldsymbol{\beta}\boldsymbol{\beta}}(\boldsymbol{\theta}) & \ddot{\ell}_{i\boldsymbol{\beta}\phi}(\boldsymbol{\theta}) \\ \ddot{\ell}_{i\phi\boldsymbol{\beta}}(\boldsymbol{\theta}) & \ddot{\ell}_{i\phi\phi}(\boldsymbol{\theta}) \end{bmatrix},$$

with

$$\ddot{\boldsymbol{\ell}}_{i\boldsymbol{\beta}\boldsymbol{\beta}}(\boldsymbol{\theta}) = \begin{cases} -\frac{1}{\phi}\Omega_i' \boldsymbol{x}_i, & i = 1, \ldots, m; \\ -\frac{2}{\phi}\left[-W_i \frac{\boldsymbol{x}_i}{\phi} + W_i'\zeta_i\right] \boldsymbol{x}_i, & i = m+1, \ldots, n; \end{cases}$$

$$\ddot{\boldsymbol{\ell}}_{i\boldsymbol{\beta}\phi}(\boldsymbol{\theta}) = \ddot{\boldsymbol{\ell}}_{i\phi\boldsymbol{\beta}}(\boldsymbol{\theta}) = \begin{cases} -\left[\frac{1}{\phi}\Omega_i' - \frac{1}{\phi^2}\Omega_i\right] \boldsymbol{x}_i, & i = 1, \ldots, m; \\ -\frac{2\zeta_i}{\phi}\left\{\left[-\frac{W_i}{\phi} + W_i'\right] - \frac{1}{\phi}W_i\right\} \boldsymbol{x}_i, & i = m+1, \ldots, n; \end{cases}$$

$$\dot{\ell}_{i\phi\phi}(\boldsymbol{\theta}) = \begin{cases} \frac{1}{\phi^2}[2\Omega_i - \phi\Omega_i']\zeta_i^{\mathrm{c}}, & i = 1, \ldots, m; \\ \frac{1}{\phi^2} + \frac{2}{\phi^2}[3W_i - \phi W_i']\zeta_i^2, & i = m+1, \ldots, n. \end{cases}$$

3.1 Statistical tests

We here consider the LR and GR statistical tests for the tobit-log-symmetric regression model. We choose these tests because they do not require the information matrix, a convenient characteristic for tobit models. Let $\boldsymbol{\theta}$ be a p-vector of parameters that index a tobit-log-symmetric model. Consider that our interest lies in test the hypothesis $\mathcal{H}_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_1^{(0)}$ against $\mathcal{H}_1 : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_1^{(0)}$, where $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^\top, \boldsymbol{\theta}_2^\top)^\top$, $\boldsymbol{\theta}_1$ is an $r \times 1$ vector of parameters of interest and $\boldsymbol{\theta}_2$ is $(p-r) \times 1$ vector of nuisance parameters.

Two popular methods for testing these linear hypotheses are by using the LR and GR test statistics, which are given by

$$\Lambda_{LR} = 2\{\ell(\widehat{\boldsymbol{\theta}}) - \ell(\widetilde{\boldsymbol{\theta}})\},$$
$$\Lambda_{GR} = \dot{\boldsymbol{\ell}}^\top(\widetilde{\boldsymbol{\theta}})(\widehat{\boldsymbol{\theta}} - \widetilde{\boldsymbol{\theta}}),$$

where $\ell(\cdot)$ is the log-likelihood function defined in (6) and $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\theta}}_1^\top, \widehat{\boldsymbol{\theta}}_2^\top)^\top$ and $\widetilde{\boldsymbol{\theta}} = (\boldsymbol{\theta}_1^{(0)\top}, \widetilde{\boldsymbol{\theta}}_2^\top)^\top$ are unrestricted and restricted ML estimators of $\boldsymbol{\theta}$, respectively. Moreover, $\dot{\boldsymbol{\ell}}(\cdot)$ is the the score vector defined in (7). In regular cases, we have that under $\mathcal{H}_0$ and $n \to \infty$, both statistical tests converge in distribution to χ_r^2. Then, $\mathcal{H}_0$ is rejected at nominal level δ if the test statistic is larger than $\chi_{1-\delta,r}^2$, the $1-\delta$ upper quantile of the χ_r^2 distribution.

3.2 Model checking

Residuals analysis are frequently used to evaluate the validity of the assumptions of the model, presence of outliers and may also be employed as tools for model selection. In the context of regression models, usually Pearson and studentized residuals are often used. Nevertheless, in a tobit scenario, these two types of residuals, even under normality, are not inadequate; see, for example, Barros et al. (2010). In the log-symmetric tobit case, we use the generalized Cox-Snell (GCS) residual given by

$$r_i^{\rm GCS} = -\log(\widehat{S}_Y(y_i;\widehat{\mu}_i,\widehat{\phi}^2,g)) = -\log(1-\widehat{F}_Y(y_i;\widehat{\mu}_i,\widehat{\phi}^2,g)), \quad i=1,\ldots,n,$$

where $\widehat{S}_Y$ denotes survival function fitted to the data. The GCS residual is unit exponential, EXP(1) in short, if the model is correctly specified whatever the specification of the model.

4 Monte Carlo simulation studies

Two MC simulation studies were carried out to evaluate the performances of the ML estimators and the statistical tests. We focus on three tobit-log-symmetric models: tobit-log-normal (tobit-LN), tobit-log-Student-t (tobit-Lt) and tobit-log-power-exponential (tobit-LPE). The `R` software was used to do all numerical calculations; see R-Team (2016).

4.1 ML estimators

A MC simulation study was carried out to evaluate the performance of the ML estimators. The study considers simulated data generated from each one of the above-mentioned models according to

$$Y_i = \begin{cases} \gamma, & Y_i^* \leq \gamma,\ i=1,\ldots,m, \\ Y_i^* = \beta_0+\beta_1 x_i+\varepsilon_i, & Y_i^* > \gamma,\ i=m+1,\ldots,n, \end{cases}$$

where ε_i is as in (4), x_i is a covariate obtained from a uniform distribution in the interval (0,1) and the true parameter values are taken as $\beta_0 = 0.2$ $\beta_1 = 0.5$. Moreover, the simulation scenario considers: sample size $n \in \{50, 100, 300, 500\}$, scale parameter $\phi \in \{1.00, 3.00, 5.00\}$, extra parameter $\xi_1 = 0.5$ (tobit-LPE), $\xi_1 = 4$ (tobit-Lt), censoring proportion $\varrho = m/n \in \{0.20, 0.50\}$, with 5,000 MC replications for each sample size.

The ML estimation results for the considered tobit-log-symmetric models are presented in Tables 2–4. The empirical bias and mean squared error (MSE) are reported. A look at the results in Tables 2–4 allows us to conclude that, for $\phi \in \{1.00, 3.00, 5.00\}$ and $\varrho \in \{0.20, 0.50\}$, as the sample size increases, the empirical bias and MSE decrease, as expected. Moreover, we note that, as the value of the parameter ϕ increases, the performance of the estimator of this parameter, deteriorates. In general, the performances of the estimators decrease when the censoring proportion increases.

Tabela 2: Empirical bias and MSE (in parentheses) from simulated data for the indicated ML estimators of the tobit-LN model parameters, n and ϱ.

n		$\varrho = 0.20$			$\varrho = 0.50$		
	ϕ	$\widehat{\phi}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$	$\widehat{\phi}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$
50	1.00	−0.0099 (0.0141)	−0.0147 (0.0918)	0.0086 (0.2715)	−0.0132 (0.0249)	−0.0212 (0.1218)	0.0087 (0.3263)
	3.00	−0.0297 (0.1269)	−0.0367 (0.8167)	0.0114 (2.4409)	−0.0394 (0.2250)	−0.0589 (1.0393)	0.0169 (2.9163)
	5.00	−0.0491 (0.3526)	−0.0589 (2.2629)	0.0144 (6.7631)	−0.0652 (0.6263)	−0.0951 (2.8684)	0.0210 (8.1019)
100	1.00	−0.0045 (0.0068)	−0.0092 (0.0441)	0.0072 (0.1308)	−0.0084 (0.0125)	−0.0094 (0.0573)	0.0060 (0.1518)
	3.00	−0.0138 (0.0610)	−0.0239 (0.3924)	0.0148 (1.1741)	−0.0257 (0.1122)	−0.0267 (0.4887)	0.0158 (1.3546)
	5.00	−0.0228 (0.1695)	−0.0370 (1.0831)	0.0191 (3.2483)	−0.0426 (0.3112)	−0.0428 (1.3439)	0.0225 (3.7660)
300	1.00	−0.0006 (0.0023)	−0.0048 (0.0147)	0.0043 (0.0428)	−0.0014 (0.0040)	−0.0053 (0.0187)	0.0040 (0.0498)
	3.00	−0.0017 (0.0209)	−0.0137 (0.1302)	0.0110 (0.3834)	−0.0041 (0.0365)	−0.0166 (0.1602)	0.0123 (0.4428)
	5.00	−0.0028 (0.0578)	−0.0222 (0.3609)	0.0171 (1.0650)	−0.0068 (0.1007)	−0.0283 (0.4382)	0.0215 (1.2277)
500	1.00	−0.0003 (0.0014)	−0.0025 (0.0088)	0.0028 (0.0258)	−0.0007 (0.0024)	−0.0016 (0.0113)	0.0003 (0.0309)
	3.00	−0.0006 (0.0127)	−0.0064 (0.0778)	0.0061 (0.2309)	−0.0011 (0.0221)	−0.0064 (0.0981)	0.0018 (0.2770)
	5.00	−0.0011 (0.0354)	−0.0105 (0.2159)	0.0104 (0.6406)	−0.0012 (0.0617)	−0.0108 (0.2697)	0.0023 (0.7681)

Tabela 3: Empirical bias and MSE (in parentheses) from simulated data for the indicated ML estimators of the tobit-Lt model parameters, n and ϱ.

n		$\varrho = 0.20$			$\varrho = 0.50$		
	ϕ	$\widehat{\phi}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$	$\widehat{\phi}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$
50	1.00	−0.0038 (0.0190)	−0.0058 (0.1148)	0.0085 (0.3473)	0.0056 (0.0347)	−0.0217 (0.1471)	−0.0181 (0.4173)
	3.00	−0.0118 (0.1706)	−0.0113 (1.0352)	0.0098 (3.1385)	0.0149 (0.3147)	−0.0464 (1.2505)	−0.0961 (3.6955)
	5.00	−0.0202 (0.4739)	−0.0196 (2.8732)	0.0287 (8.7144)	0.0252 (0.8749)	−0.0715 (3.4329)	−0.0308 (9.2639)
100	1.00	0.0011 (0.0102)	−0.0051 (0.0560)	0.0050 (0.1668)	0.0031 (0.0185)	−0.0154 (0.0719)	0.0115 (0.1958)
	3.00	0.0023 (0.0911)	−0.0111 (0.5021)	0.0079 (1.4986)	0.0107 (0.1682)	−0.0409 (0.6101)	0.0218 (1.7335)
	5.00	0.0037 (0.2529)	−0.0173 (1.3919)	0.0197 (4.1527)	0.0176 (0.4673)	−0.0640 (1.6827)	0.0286 (4.8195)
300	1.00	−0.0005 (0.0033)	−0.0027 (0.0181)	0.0047 (0.0547)	0.0003 (0.0059)	−0.0069 (0.0229)	0.0083 (0.0636)
	3.00	−0.0018 (0.0294)	−0.0073 (0.1622)	0.0123 (0.4910)	0.0007 (0.0536)	−0.0168 (0.1941)	0.0180 (0.5621)
	5.00	−0.0027 (0.0814)	−0.0117 (0.4497)	0.0109 (1.3623)	0.0022 (0.1494)	−0.0285 (0.5374)	0.0292 (1.5663)
500	1.00	−0.0004 (0.0019)	−0.0010 (0.0108)	0.0011 (0.0331)	0.0001 (0.0035)	−0.0024 (0.0134)	0.0013 (0.0379)
	3.00	−0.0010 (0.0173)	−0.0023 (0.0972)	0.0019 (0.2977)	0.0005 (0.0316)	−0.0037 (0.1155)	0.0021 (0.3394)
	5.00	−0.0018 (0.0482)	−0.0038 (0.2699)	0.0031 (0.8267)	0.0002 (0.0882)	−0.0067 (0.3167)	0.0037 (0.9394)

4.2 Statistical tests

We now present a MC simulation study to evaluate and compare the performance of the LR and GR tests. We consider again the following models: tobit-LN, tobit-Lt and tobit-LPE. The simulation scenario considers: sample size $n \in \{50, 100, 300, 500\}$, scale parameter $\phi = 3.00$, extra parameter $\xi_1 = 0.5$ (tobit-LPE), $\xi_1 = 5$ (tobit-Lt), censoring proportion $\varrho = m/n \in \{0.3, 0.5\}$, with 5,000 MC replications for each sample size. We consider as data generating process the model

$$Y_i = \begin{cases} \gamma, & Y_i^* \leq \gamma, \; i = 1, \ldots, m, \\ Y_i^* = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} + \beta_4 x_{4i} + \varepsilon_i, & Y_i^* > \gamma, \; i = m+1, \ldots, n, \end{cases}$$

where ε_i is as in (4), with $\beta_0 = 1.0$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\beta_3 = 0.8$ and $\beta_4 \in \{-1.00, -0.75, -0.25$ $0.00, 0.25, 0.75, 1.00\}$. The covariate values were taken as random draws from the U(0,1) distribution. The interest lies in testing $\mathcal{H}_0 : \beta_4 = 0$ against $\mathcal{H}_1 : \beta_4 \neq 0$.

Tables 5-7 present the simulation results regarding the powers of the tests, namely, their capacity

Tabela 4: Empirical bias and MSE (in parentheses) from simulated data for the indicated ML estimators of the tobit-LPE model parameters, n and ϱ.

n		$\varrho = 0.20$			$\varrho = 0.50$		
	ϕ	$\widehat{\phi}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$	$\widehat{\phi}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$
50	1.00	−0.0090 (0.0194)	−0.0064 (0.1903)	0.0087 (0.5807)	−0.0109 (0.0314)	−0.0282 (0.2277)	0.0228 (0.6642)
	3.00	−0.0272 (0.1742)	−0.0151 (1.7096)	0.0187 (5.2207)	−0.0312 (0.2793)	−0.0653 (1.9717)	0.0269 (5.9557)
	5.00	−0.0456 (0.4837)	−0.0237 (4.7496)	0.0285 (9.5076)	−0.0728 (1.5164)	−0.1393 (5.5873)	0.0391 (9.4605)
100	1.00	−0.0047 (0.0099)	−0.0028 (0.0910)	0.0002 (0.2815)	−0.0037 (0.0159)	−0.0149 (0.1077)	0.0106 (0.3170)
	3.00	−0.0061 (0.0888)	−0.0081 (0.8171)	0.0142 (2.5290)	−0.0122 (0.1427)	−0.0339 (0.9188)	0.0174 (2.7931)
	5.00	−0.0102 (0.2469)	−0.0135 (2.2718)	0.0048 (7.0339)	−0.0205 (0.3969)	−0.0537 (2.5311)	0.0170 (7.7414)
300	1.00	−0.0021 (0.0032)	−0.0019 (0.0295)	0.0002 (0.0865)	−0.0010 (0.0052)	−0.0063 (0.0341)	0.0043 (0.0956)
	3.00	−0.0026 (0.0287)	−0.0051 (0.2653)	0.0030 (0.7783)	−0.0069 (0.0468)	−0.0172 (0.2901)	0.0130 (0.8346)
	5.00	−0.0047 (0.0798)	−0.0078 (0.7366)	0.0021 (2.1619)	−0.0097 (0.1299)	−0.0253 (0.7998)	0.0047 (2.3190)
500	1.00	−0.0008 (0.0019)	−0.0017 (0.0177)	0.0001 (0.0525)	−0.0003 (0.0031)	−0.0005 (0.0204)	0.0033 (0.0574)
	3.00	−0.0021 (0.0176)	−0.0047 (0.1593)	0.0018 (0.4733)	−0.0017 (0.0280)	−0.0033 (0.1730)	0.0030 (0.5050)
	5.00	−0.0011 (0.0176)	−0.0051 (0.1593)	0.0014 (0.4733)	−0.0079 (0.0775)	−0.0162 (0.4754)	0.0031 (1.3952)

to identify a false null hypothesis. Note, however, that we also consider the case where the null hypothesis is true ($\beta_4 = 0.00$ in the data generation). From Tables 5-7, we observe that the power associated with the LR and GR tests increases as a function of the sample size, as expected. We also observe that the power of the tests decreases when the censoring proportion increases. In general, the results show that both tests have similar power.

5 Application

Tobit-log-symmetric models are now used to analyse a data set from a case-study of measles vaccines, corresponding to antibody concentration levels (response variable, Y_i) collected from 330 children at 12 months of age; see Moulton and Halsey (1995). In the measurement of antibody concentration by quantitative assays, there is always a concentration value, γ say, below which an exact measurement cannot be computed, independently of the employed technique. Then, this value γ can be used to substitute a value for the censored observation. In the measles vaccine data, the value of γ was 0.1 international units (IU) or -2.306 in logarithm scale. It was verified that 86 (26.1%) of the observations fell below γ and then were recorded as 0.1. The covariates considered in the study were: x_{i1} is the type of vaccine used (0 if Schwartz and 1 if Edmonston-Zagreb); x_{i2} is the level of the dosage (0 if medium and 1 if high); and x_{i3} is the gender where 0 is male and 1 is female.

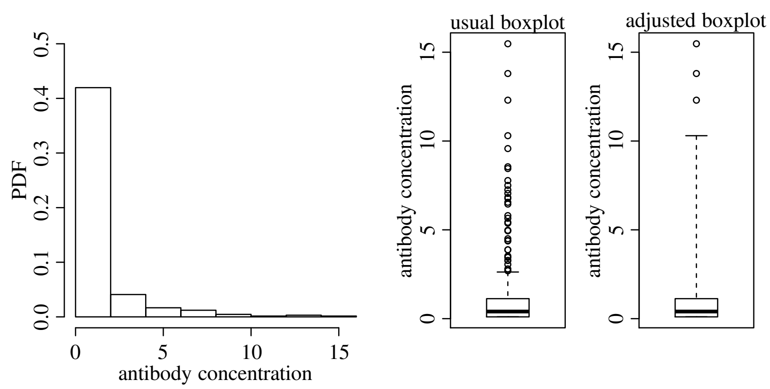
Table 8 reports descriptive statistics of the observed antibody concentration levels, including the median (MD), mean ($\overline{y}$), standard deviation (SD), coefficient of variation (CV), skewness (CS) and kurtosis (CK), and minimum ($y_{(1)}$) and maximum ($y_{(n)}$) values. From this table, note the right skewed nature and high kurtosis level of the data distribution.

Figure 1 presents the histogram and boxplots for the measles vaccine data. Note that the skewness observed in Table 8 is confirmed by the histogram shown in Figure 1(a). The adjusted boxplot for the measles vaccine data indicates that some potential outliers identified by the usual boxplot are not outliers; see Figure 1(b). The adjusted boxplot is used when the data is skew distributed; see Hubert and Vandervieren (2008).

We now analyse the measles vaccine data using the tobit-log-symmetric model, which can be

Tabela 5: Power study (%) for different values of β_4 and models (nominal level = 1%).

		tobit-LN				tobit-Lt				tobit-LPE			
		$\varrho = 0.20$		$\varrho = 0.50$		$\varrho = 0.20$		$\varrho = 0.50$		$\varrho = 0.20$		$\varrho = 0.50$	
	β_4	LR	GR	LR	GR	LR	GR	LR	GR	LR	GR	LR	GR
n 50	-1.00	4.04	3.18	3.38	2.62	4.00	3.16	3.54	2.92	4.26	3.38	3.66	3.18
	-0.75	2.90	2.38	2.56	1.94	2.88	2.48	2.82	2.28	3.24	2.48	2.82	2.52
	-0.25	1.86	1.34	1.58	1.20	2.28	1.60	2.12	1.62	2.44	1.72	2.28	1.92
	0.00	1.60	1.24	1.46	1.16	1.88	1.40	1.86	1.48	2.04	1.62	2.08	1.60
	0.25	1.76	1.42	1.74	1.24	2.02	1.62	1.92	1.50	2.36	1.82	2.20	1.78
	0.75	2.74	2.06	2.56	2.06	3.00	2.14	2.58	2.28	3.16	2.44	2.84	2.38
	1.00	3.48	2.80	3.36	2.78	3.62	3.06	3.40	2.96	3.98	3.10	4.06	3.20
100	-1.00	5.60	5.08	4.84	4.34	5.82	5.36	5.28	4.80	6.50	5.60	5.60	5.08
	-0.75	3.52	3.30	3.28	2.96	3.62	3.30	3.58	3.28	4.58	3.84	3.84	3.50
	-0.25	1.42	1.24	1.44	1.26	1.76	1.54	1.72	1.62	1.96	1.70	1.98	1.84
	0.00	1.12	1.02	1.34	1.14	1.38	1.26	1.54	1.44	1.68	1.42	1.78	1.64
	0.25	1.40	1.24	1.46	1.36	1.82	1.50	1.72	1.54	1.98	1.56	1.84	1.64
	0.75	2.90	2.60	2.84	2.50	3.58	3.16	3.14	2.84	3.82	3.40	3.40	3.30
	1.00	4.72	4.18	4.20	3.82	4.98	4.62	4.62	4.36	5.80	5.26	4.98	4.62
300	-1.00	15.52	15.20	12.90	12.52	16.54	16.20	13.78	13.66	17.34	16.90	14.62	14.50
	-0.75	7.90	7.60	6.56	6.44	8.48	8.22	7.24	7.00	9.62	9.18	7.84	7.62
	-0.25	1.64	1.60	1.52	1.42	1.84	1.76	1.82	1.72	2.16	2.04	1.84	1.84
	0.00	1.10	0.98	1.18	1.12	1.44	1.38	1.41	1.40	1.70	1.62	1.80	1.64
	0.25	1.78	1.74	1.80	1.78	1.88	1.74	2.08	1.98	2.14	2.02	2.20	2.18
	0.75	8.54	8.14	7.42	7.24	9.00	8.70	8.12	7.84	9.84	9.32	8.32	7.92
	1.00	16.34	16.06	13.68	13.14	16.88	16.46	14.08	13.78	17.54	16.96	14.74	14.30
500	-1.00	29.58	29.12	23.40	23.04	29.34	29.16	23.74	23.46	29.90	29.50	24.28	24.04
	-0.75	13.72	13.56	11.30	11.12	14.08	13.74	11.90	11.72	15.20	14.94	12.74	12.42
	-0.25	2.00	1.94	1.94	1.86	2.44	2.40	2.34	2.32	3.00	3.02	2.62	2.60
	0.00	1.08	1.08	1.04	1.02	1.30	1.22	1.16	1.10	1.86	1.80	1.64	1.62
	0.25	2.02	1.98	1.90	1.82	2.46	2.36	2.24	2.22	2.80	2.60	2.38	2.40
	0.75	15.24	15.16	12.82	12.52	15.44	15.32	12.84	12.80	16.38	16.04	13.46	13.20
	1.00	29.84	29.52	24.56	24.20	30.26	29.98	25.12	24.88	30.90	30.16	25.12	25.00



(a) Histogram (b) Boxplots

Figura 1: Histogram and boxplots for the measles vaccine data.

written as

$$Y_i = \begin{cases} 0.1, & Y_i^* \leq 0.1, \; i = 1, \ldots, 85, \\ Y_i^* = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{2i} + \beta_3 x_{3i} + \varepsilon_i, & Y_i^* > 0.1, \; i = 86, \ldots, 330, \end{cases}$$

Tabela 6: Power study (%) for different values of β_4 and models (nominal level = 5%).

		tobit-LN				tobit-Lt				tobit-LPE			
		$\varrho = 0.20$		$\varrho = 0.50$		$\varrho = 0.20$		$\varrho = 0.50$		$\varrho = 0.20$		$\varrho = 0.50$	
	β_4	LR	GR	LR	GR	LR	GR	LR	GR	LR	GR	LR	GR
n 50	−1.00	11.58	10.76	10.98	10.06	12.82	11.96	11.76	10.92	13.66	12.46	12.40	11.52
	−0.75	9.84	9.14	9.22	8.54	10.52	9.94	9.92	9.30	11.50	10.24	10.58	9.82
	−0.25	7.58	6.84	7.06	6.48	7.74	7.12	7.90	6.92	8.56	7.60	8.38	7.86
	0.00	7.08	6.66	6.78	6.00	7.60	6.74	7.12	6.44	8.04	7.16	7.82	7.14
	0.25	7.42	6.76	6.74	6.16	7.72	7.22	7.22	6.80	8.48	7.42	8.16	7.30
	0.75	8.90	8.26	8.60	7.82	9.60	8.84	9.04	8.48	10.94	9.66	10.08	9.30
	1.00	10.66	9.92	9.90	9.30	11.50	10.42	10.76	10.04	12.28	11.44	11.72	10.88
100	−1.00	16.30	15.76	14.32	14.02	16.88	16.30	15.06	14.68	17.78	17.28	15.44	15.18
	−0.75	11.36	11.04	10.94	10.64	12.44	11.82	11.52	11.12	13.34	12.54	12.36	11.86
	−0.25	6.72	6.66	6.64	6.28	7.22	6.74	6.92	6.58	8.02	7.56	7.44	7.20
	0.00	5.36	5.14	5.56	5.30	6.30	5.90	6.30	5.98	7.36	6.68	6.26	6.18
	0.25	5.70	5.42	5.78	5.54	6.50	6.26	6.72	6.32	7.44	7.00	7.10	6.70
	0.75	10.94	10.54	10.06	9.60	11.84	11.28	10.72	10.48	12.64	11.82	11.70	11.30
	1.00	15.42	14.86	13.46	13.02	16.32	15.64	14.14	13.78	16.82	15.92	14.98	14.54
300	−1.00	34.22	33.94	30.08	29.80	35.06	34.84	30.84	30.64	34.98	34.70	31.30	31.30
	−0.75	21.94	21.78	19.16	18.90	22.56	22.40	20.20	20.12	23.38	23.14	20.68	20.72
	−0.25	6.46	6.22	6.74	6.66	7.40	7.28	7.18	7.12	8.62	8.50	7.80	7.70
	0.00	5.30	5.28	5.28	5.22	5.62	5.54	5.68	5.58	6.18	6.10	6.18	6.12
	0.25	6.78	6.70	7.06	6.92	7.60	7.40	7.68	7.54	8.60	8.50	8.08	8.00
	0.75	22.56	22.46	19.96	19.62	22.80	22.58	20.12	19.90	24.04	23.70	20.80	20.42
	1.00	35.04	34.60	31.22	30.92	35.38	35.22	31.24	31.02	36.12	36.08	32.18	32.08
500	−1.00	53.32	53.18	47.88	47.68	52.94	52.74	47.68	47.60	52.50	52.36	47.38	47.38
	−0.75	33.12	32.96	28.62	28.50	33.08	32.96	28.70	28.46	33.64	33.38	29.34	29.14
	−0.25	7.82	7.78	7.36	7.30	8.64	8.54	8.24	8.14	9.46	9.42	8.58	8.60
	0.00	5.40	5.40	5.22	5.10	5.72	5.64	5.72	5.62	6.68	6.50	6.26	6.22
	0.25	8.16	8.12	7.98	7.78	8.90	8.86	8.52	8.42	9.54	9.36	9.16	9.04
	0.75	33.86	33.78	29.16	29.02	34.42	34.26	30.10	29.94	34.48	34.54	29.90	29.88
	1.00	53.10	53.06	46.98	46.78	52.96	52.94	47.24	47.08	52.70	52.58	47.44	47.26

where $\varepsilon_i \stackrel{\text{IID}}{\sim} \text{S}(0, 1, g)$. In addition to the tobit-log-symmetric models studied in the simulation study, we also consider the tobit-Birnbaum-Saunders (tobit-BS) and tobit-Birnbaum-Saunders-t (tobit-BS-t) models.

Table 9 reports the ML estimates, computed by the BFGS quasi-Newton method, SEs and Akaike (AIC) and Bayesian information (BIC) criteria. For comparison, the results of the classical tobit-NO model (Tobin, 1958) showed in Equation (1), are given as well. From Table 9, note that, all the tobit-log-symmetric models provide better adjustments compared to the tobit-NO model based on the values of AIC and BIC. Particularly, the tobit-LN has the lowest AIC and BIC values.

Figure 2 displays the quantile versus quantile (QQ) plots with simulated envelope of the GCS residuals for the tobit-NO, tobit-LN, tobit-Lt, tobit-LPE, tobit-BS and tobit-BS-t models. This figure indicates that GCS residuals in the tobit-log-symmetric models (except the tobit-LPE) show better agreements with the EXP(1) distribution. In special, observe a quite good agreement in the tobit-BS case and a poor agreement in the tobit-NO case.

Next, we test the null hypotheses a) $\mathcal{H}_0 : \beta_1 = 0$, b) $\mathcal{H}_0 : \beta_2 = 0$ and c) $\mathcal{H}_0 : \beta_3 = 0$, using the LR and GR tests. We consider only the tobit-LN model as it has presented the lowest AIC and BIC values. The corresponding LR and GR tests p-values are: a) 0.0970 (LR) and 0.0975 (GR); b) 0.4656 (LR) and 0.4657 (GR); c) 0.6446 (LR) and 0.6448 (GR).

Tabela 7: Power study (%) for different values of β_4 and models (nominal level = 10%).

		tobit-LN				tobit-Lt				tobit-LPE			
		$\varrho = 0.20$		$\varrho = 0.50$		$\varrho = 0.20$		$\varrho = 0.50$		$\varrho = 0.20$		$\varrho = 0.50$	
	β_4	LR	GR	LR	GR	LR	GR	LR	GR	LR	GR	LR	GR
n 50	-1.00	18.92	18.28	18.06	17.54	20.06	19.46	19.52	18.78	21.02	19.86	20.00	19.64
	-0.75	16.50	16.00	16.24	15.52	17.46	16.76	17.40	16.68	18.40	17.44	17.98	17.30
	-0.25	13.42	12.94	12.98	12.36	14.54	13.94	13.54	12.96	15.76	14.40	14.68	13.92
	0.00	12.68	12.20	12.58	11.92	13.66	13.20	13.22	12.36	15.20	14.14	14.26	13.46
	0.25	12.88	12.42	12.68	12.02	14.00	13.22	13.38	12.84	14.88	14.28	14.42	13.54
	0.75	15.54	14.88	14.48	13.76	16.50	15.68	15.72	15.18	17.40	16.18	16.76	16.20
	1.00	18.06	17.44	16.66	16.02	19.18	18.56	17.92	17.24	20.20	18.90	18.60	17.88
100	-1.00	24.26	23.70	22.84	22.54	25.08	24.90	23.10	22.86	25.98	25.12	23.62	23.40
	-0.75	19.06	18.82	17.86	17.28	20.10	19.80	18.50	18.32	20.80	20.24	18.74	18.76
	-0.25	11.98	11.80	11.90	11.62	13.00	12.74	12.68	12.48	14.46	13.76	13.84	13.32
	0.00	11.22	10.90	11.50	11.20	11.70	11.52	12.14	12.00	12.88	12.62	12.82	12.58
	0.25	11.80	11.50	11.88	11.52	12.40	12.16	12.66	12.36	14.10	13.42	13.78	13.58
	0.75	18.36	17.94	17.56	17.04	19.94	19.48	18.06	17.90	20.94	20.18	18.82	18.52
	1.00	24.56	24.02	22.48	21.92	25.38	25.00	23.06	22.64	26.48	26.00	23.70	23.20
300	-1.00	46.70	46.66	42.04	41.90	46.56	46.48	42.02	41.98	46.52	46.50	42.78	42.64
	-0.75	32.12	31.98	29.26	29.06	32.90	32.74	30.24	30.16	33.34	32.94	30.60	30.52
	-0.25	12.76	12.68	11.78	11.76	13.80	13.66	13.68	13.62	15.06	14.66	14.62	14.54
	0.00	9.58	9.54	10.34	10.26	10.74	10.68	11.28	11.20	12.62	12.62	12.82	12.58
	0.25	13.26	13.22	12.68	12.58	13.76	13.62	13.28	13.26	14.60	14.48	14.48	14.22
	0.75	32.60	32.46	29.94	29.74	33.40	33.30	30.38	30.18	34.14	33.68	31.04	30.70
	1.00	47.20	47.08	43.34	43.04	47.40	47.28	43.36	43.24	47.16	47.04	43.86	43.54
500	-1.00	64.84	64.78	59.90	59.82	65.06	65.02	59.90	59.78	64.18	63.96	59.92	59.88
	-0.75	45.82	45.74	41.26	41.12	45.40	45.30	41.22	41.14	46.02	45.92	41.28	41.20
	-0.25	13.62	13.56	13.04	12.92	14.26	14.24	14.14	13.98	15.78	15.58	14.92	14.76
	0.00	10.66	10.04	10.76	10.68	10.84	10.84	10.97	10.90	12.32	12.22	12.04	11.90
	0.25	14.80	14.68	14.12	14.04	15.34	15.30	15.12	15.08	16.66	16.46	15.70	15.70
	0.75	45.98	45.90	41.46	41.40	45.46	45.36	41.16	41.08	45.82	45.86	41.84	41.84
	1.00	65.68	65.66	60.16	60.12	65.58	65.54	60.14	60.00	65.64	65.24	59.74	59.86

Tabela 8: Summary statistics for the measles vaccine data.

MD	$\bar{y}$	MD	SD	CV	CS	CK	$y_{(1)}$	$y_{(n)}$	n
0.4	1.20	0.40	2.10	174.74%	3.46	14.37	0.10	15.47	330

Tabela 9: ML estimates (with SE in parentheses) and AIC values for the indicated models with the measles vaccine data

Model	AIC	BIC	ϕ	ξ_1	ξ_2	β_0	β_1	β_2	β_3
tobit-NO	1299.27	1318.27	0.945			0.597	0.225	-0.228	0.271
			(0.047)			(0.288)	(0.297)	(0.295)	(0.296)
tobit-LN	1122.28	1141.28	1.666			-1.239	0.315	0.138	0.087
			(0.080)			(0.184)	(0.190)	(0.189)	(0.189)
tobit-Lt	1130.68	1153.47	1.474	5		-1.207	0.319	0.208	0.077
			(0.081)			(0.183)	(0.189)	(0.188)	(0.189)
tobit-LPE	1123.67	1146.47	1.311	0.3		-1.182	0.260	0.178	0.070
			(0.070)			(0.173)	(0.180)	(0.175)	(0.181)
tobit-BS	1168.38	1187.37		1.545		-0.910	0.178	0.073	0.121
				(0.081)		(0.105)	(0.127)	(0.126)	(0.126)
tobit-BS-t	1126.16	1148.96		1.662	4	-1.241	0.305	0.086	0.113
				(0.102)		(0.186)	(0.191)	(0.190)	(0.190)

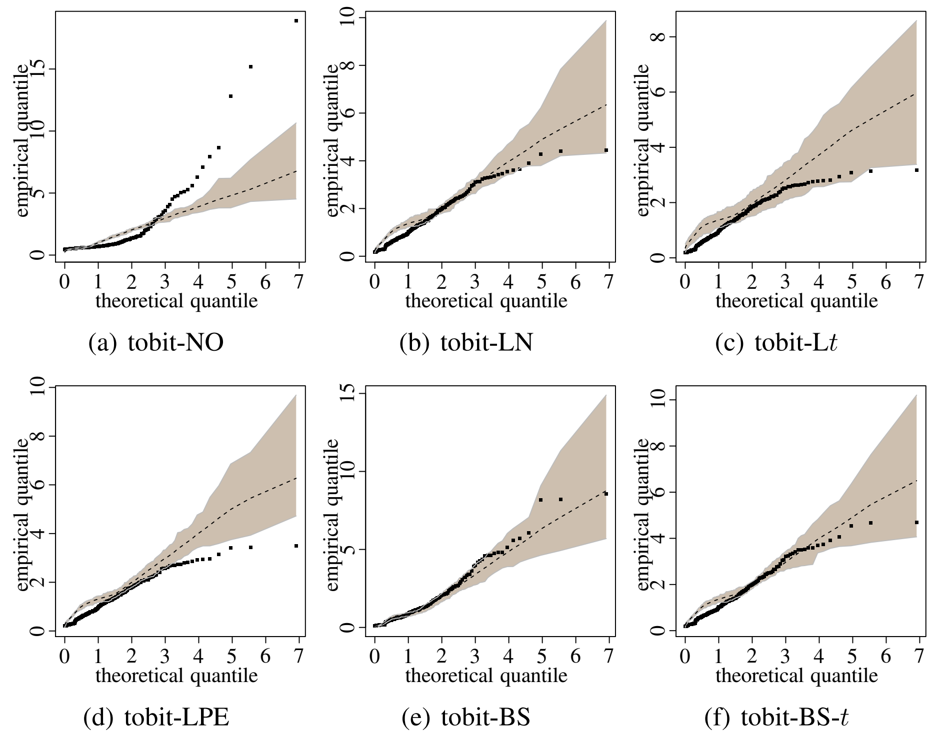


Figura 2: QQ plot and its envelope for the GCS residual for the tobit-log-symmetric models with measles vaccine data.

6 Concluding remarks

We have proposed and analyzed a new class of tobit models for left-censored data. We have considered a likelihood-based approach for parameter estimation. We have addressed the issue of performing testing inference in the proposed class of tobit models by using the likelihood ratio and gradient statistics. Monte Carlo simulations studies were carried out to evaluate the behaviour of the maximum likelihood estimators and the likelihood ratio and gradient tests. We have applied the proposed models to a real-world data set of measles vaccine data in Haiti. The application has favored the use of tobit-log-symmetric models over the classical tobit-normal model.

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