Título: Efficiency in Two Player Repeated Games of Imperfect Monitoring

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Resumo:

A literatura em jogos repetidos com monitoramento público imperfeito tem se concentrado em eficiência aproximada, atingida conforme os jogadores se tornam ifinitamente pacientes.

Esse trabalho, por outro lado, estuda os equilíbrios eficientes com jogadores impacientes.

É mostrado que, no caso de dois jogadores, uma caracterização completa do conjunto de equilíbrios eficientes em termos simples é possível. E que eficiência depende da intensidade de punições necessárias para implementar certas ações.

Como exemplo, uma demonstração de ineficiência no dilema dos prisioneiros é dada, sob hipóteses mais fracas do que o ususal.

Abstract:

Most of the literature on repeated games with imperfect public monitoring has focused on approximate efficiency, achieved as players become infinitely patient.

By contrast, this note studies the set of fully efficient public equilibria with impatient players.

It is shown that, for two player games, a full characterization of this set can be given in simple terms. And that efficiency depends on the severity of punishments needed to enforce certain actions.

As an example, a proof of inefficiency in the prisoner's dilemma is given, under scarce assumptions.

Palavras Chave: Jogos repetidos, monitoramento público imperfeito, equilíbrio perfeito público, eficiência.

Key Words: Repeated games, imperfect public monitoring, public perfect equilibrium, efficiency.

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Efficiency in Two Player Repeated Games of Imperfect Monitoring

ABSTRACT. Most of the literature on repeated games with imperfect public monitoring has focused on approximate efficiency, achieved as players become infinitely patient.

By contrast, this note studies the set of fully efficient public equilibria with impatient players.

It is shown that, for two player games, a full characterization of this set can be given in simple terms. And that efficiency depends on the severity of punishments needed to enforce certain actions.

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Classic results in game theory guarantee that, essentially, any outcome is possible in infinitely repeated games. Since players may be harshly (and eternally) punished for any deviation from equilibria, efficient outcomes can always be enforced. Or so the folk theorem with complete information says. But if monitoring is imperfect – as it often is – such conclusions may go adrift.

Here we study games in which players can not perfectly observe each other's actions. Only an imperfect public signal is available. This class of games is interesting because (i)It has diverse applications, new and old, including relational contracts (Levin 2003), monetary policy (Athey et. al. 2005), collusion (Green and Porter 1984) and bidding rings (Aoyagi 2003). (ii)Little is known about their efficient equilibria. In fact, the first example of such efficient equilibria was given in Athey and Bagwell (2001).

This note shows this result may be extended to generic two player games of imperfect public monitoring. A complete characterization of the efficient equilibria of such games is given.

A related result is Fudenberg et. al. (2006) which present an algorithm for finding the limit set of efficient equilibria, as players become infinitely patient. Yet, their work focuses on the n-player case, which admits considerably less structure.

By contrast, lemma 2 describes this set for any given discount factor. It is also shown that, if players are sufficiently patient this entire limit set is achieved. Lemma 1 relates the shape (and all important non-emptiness) of this set and the severity of punishments needed to enforce certain actions. A proof of inefficiency in the prisoner's dilemma, under scarce assumptions, follows easily.

1. The Model

1.1. The setup and basic results. Two players play a stage game at $t = 0, 1, 2, \cdots$. They take actions a_i in a *finite* set A_i. Actions are not directly observable, but they induce probabilities $\pi(\cdot|a)$ on a *finite* set of public outcomes Y. Moreover we assume that these probabilities have a *constant support*, not depending on a[1]. The actual payoff $r_i(a_i, y)$ to a player depends on his own action and on the public outcome. But not directly on the other player's action (although does affect the distribution of y).

Example 1 (Partners' dillema). *The two players are owners of a firm, and operate it. Profits y are random, but their distribution will depend on effort levels* $a_i \in A_i = \{work, shirk\}$. *Their payoffs are* r_i *=profit/2−effort.*

Let $g_i(a) = E(r_i(a_i, y)|a)$ be the average gain from playing a profile a.

A typical stage game has an strategic form such as[2]

$$\begin{array}{cc} & \begin{array}{cc} c & n \end{array} \\ \begin{array}{c} c \\ n \\ f \end{array} & \begin{pmatrix} A & B \\ C & D \\ E & F \end{pmatrix} \end{array}$$

Throughout this note we will use letters $A, B, C, \cdots$ to denote both action profiles, as cc and their payoff vectors $g(cc)$.

We admit correlated equilibria, as in Aumann (1987). That is, players can condition their actions on a public randomization device. This makes V, the set of feasible payoffs of the stage game a polygon - the convex closure of points $A, B, C \cdots$ (figure 1). Assume, also, that each side contains only two pure action profiles[3].

In the repeated game players maximize

$$v_i = (1 - \delta) \sum \delta^t E g_i^t$$

for $0 \leq \delta < 1$. The factor $(1 - \delta)$ normalizes supergame payoffs as average payoffs. So the set of feasible payoffs of repeated play is also V.

Our solution concept is perfect public equilibrium. That is, subgame perfect equilibria in which players condition their actions only on the public history. This is a widely used solution concept, and precise definitions can be found in Fudenberg and Tirole (1991) or Mailath (forthcoming). Abreu et. al. (1990) show that, as long as other players play public strategies, there is no gain in conditioning on

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[1] The assumptions of finite A and Y are included to avoid pathologies.The results carry over to most cases where they are not satisfied. In fact, Athey and Bagwell (2001), who provided the original example of efficient equilibria, studied a model with a continuum of actions. The constant support hypothesis means that a player can never be certain about the other's actions. It is assumed, for instance, by Abreu et. al. (1990). Without it, efficient equilibria may prescribe off the equilibrium path inefficient punishments. So a full characterization of efficient values would depend on knowledge of the whole set of equilibria, which is largely unavailable. Still, as players become infinitely patient, Fudenberg et. al. (2006) show much can be said.

The constant support assumtion is the reason we said our result hold only generically.

[2] We will use $A, B, C, \cdots$ to denote both action profiles, as cc and their payoff vectors $g(cc)$.

[3] More precisely, only payoff vectors of two pure action profiles.

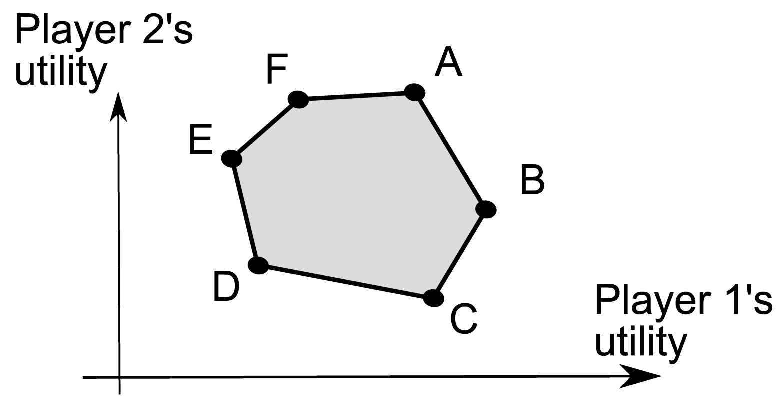


FIGURE 1. A feasible set.

private history[4]. Let $PPE(\delta)$ denote the set of perfect public equilibria for a given discount factor δ.

We are now turn to the efficient equilibrium values - those that are not Pareto dominated by any feasible payoff. Such a value must be located on a downward sloping side of the polygon. From now on we consider such a side, $\overline{AB}$ with slope m[5].

PROPOSITION 1. *For any given $\delta < 1$, the set of efficient equilibria in $\overline{AB}$ is a (possibly empty) compact interval $I(\delta)$. As δ nears 1, these intervals stabilize - there exists $\overline{\delta}$ such that $I(\delta) = I(\overline{\delta})$, for all $\delta \geq \overline{\delta}$.*

PROOF. $PPE(\delta)$ is compact and convex[6], so $I(\delta) = PPE(\delta) \cap \overline{AB}$ must be a closed interval. The fact that $I(\delta)$ is constant for large δ is a direct consequence of lemma 1. □

This remark draws a qualitative picture of how the efficient equilibria $I(\delta)$ evolve. Still, it does not speak on their shape, or even on the possibility of efficiency. The following lemma does so, in terms of the severity of punishment needed to enforce action profiles A and B.

LEMMA 1. *Let a and b be points in $\overline{AB}$ with $a_1 = A_1 + P_A^1$ and $b_2 = B_2 + P_B^2$(figure 2). For large δ the set of efficient equilibria in $\overline{AB}$ is*

- *$[a, b]$ if $a_1 < b_2$.*
- *The static equilibria, otherwise.*

Naturally, without defining the numbers P, the lemma does not say much. Informally, the number P_A^1 is the minimum level of punishment that has to be inflicted on player 1 to play his ill-favored action A. A precise definition will be given shortly.

[4]Yet, Kandori and Obara (2006) show that efficiency may improve if all players use private strategies.

[5]For simplicity assume $A_1 < B_1$ and $m \neq 0$ or ∞.

[6]Abreu et. al. show it to be compact, and it is convex for we are assuming correlated equilibria.

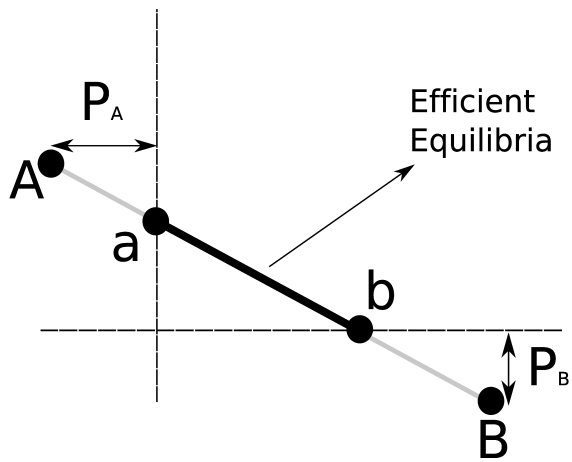


FIGURE 2. The efficient equilibria of lemma 1.

This lemma has an interesting corollary. Suppose player 2 plays the same action in profiles A and B. So player 1 can deviate from A to B, and the punishment $P_A^1 \geq B_1 - A_1$. But then the lemma implies there are no efficient equilibria in $\overline{AB}$:

COROLLARY 1. *If a player uses the same action in profiles A and B, and neither is a static equilibrium, then there are no efficient equilibria in $\overline{AB}$.*

The following examples illustrate the techniques.

1.2. Partners' dilemma. Let w be work and s be shirk. We will use ww, ws, sw and ss to denote both action profiles and their payoff vectors. For the partners' dilemma to be interesting, payoffs are usually distributed as in the classic prisoner's dilemma. That is,

$$
\begin{aligned}
ws_1 &< ss_1 < ww_1 < sw_1 \\
sw_2 &< ss_2 < ww_2 < ws_2
\end{aligned} \tag{1.1}
$$

Mailath and Samuelson (forthcoming) and Radner et. al. (1986) give examples of such games which are bounded away from efficiency. But Fudenberg et. al. (1994) show that as δ nears 1, these games may have approximately efficient equilibria In fact, their folk theorem holds generically if $\#Y \geq 4$. And, to our knowledge, the literature is silent about inefficiency in this case.

Yet, efficient equilibria could only be achieved on the sides $\overline{(ws)(ww)}$ or $\overline{(sw)(ww)}$ (figure 3). But then corollary 1 shows they can never achieve exactly efficient equilibria, irrespective of the signal space Y.

PROPOSITION 2. *Consider a discounted infinitely repeated game G with imperfect public monitoring and*

- *Two stage actions for each player, w and s.*
- *Stage payoffs respecting 1.1.*
- *Constant support.*

then G has no efficient perfect public equilibrium.

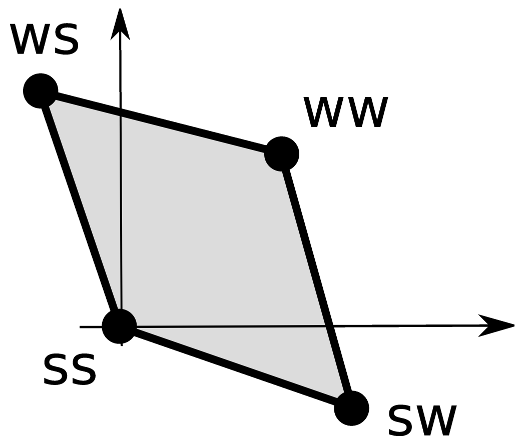
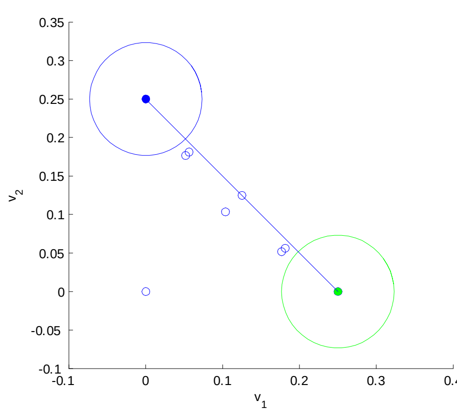
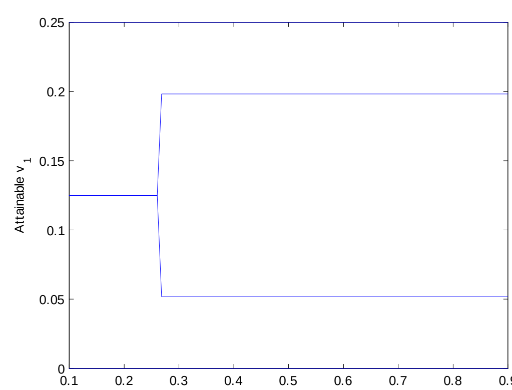


Figure 3. The prisoner's dilemma.

Still, there is a similar example in which efficiency is possible. Suppose each partner can choose an effort level in $E = \{0, 1/8, 1/4\}$. The firm's average profits are given by $\sqrt{\text{effort}}$, so the efficient level of effort is $1/4$. In this case, partners can take turns working $1/4$. So they can arrange an incentive scheme in which getting good results today means a lower probability of working tomorrow, and efficiency may be achieved. A numerical example is shown in figure 1.2. The points are pure strategy payoffs. Efficient equilibria are the points on the line outside the circles. Figure 1.2 shows how the interval of efficient equilibria evolves as δ grows.



The extended partership game.



Attainable efficient values, for given δ.

1.3. Efficiency and punishments. We now define the quantities mentioned in lemma 1.

Let $u = (u_1, u_2) : Y \longrightarrow \mathbb{R}^2$ be reward functions. If the static game were to be played once, and payments u to be made conditional on Y, α would be an equilibrium iff

$$\begin{array}{rlll} & g_i(\alpha) & + & E(u_i(y)|\alpha) \\ \geq & g_i(a_i,\alpha_{-i}) & + & E(u_i(y)|a_i,\alpha_{-i}) \quad \text{for } i=1,2 \text{ and every } a_i \text{ in } A_i \end{array} \tag{1.2}$$

We will say that the reward function u implements α if 1.2 holds.

P_A^1 will now be defined in terms of an incentive problem, of implementing A with two conditions. First, there is a weighted budget balance: player 2 always receives $m*$what player 1 receives. And secondly, the expected payment to each player is 0. Then P_A^1 is the minimum, over all transfer functions, of the largest punishment that has to be inflicted on player 1. More precisely:

Definition 1. *Let P_A^1 be*

$$\underset{u}{\text{Min}}(\underset{y}{\text{Max}} -u_1(y)) \tag{1.3}$$

$$s.t. \begin{cases} u \text{ implements } A & \\ u_2 = mu_1 & \text{(budget balance)} \\ E(u(y)|A) = 0 & \text{(0 expected payment)} \end{cases}$$

Likewise, P_B^2 is defined. There is an alternative definition of P_A^1 through a simpler, albeit less meaningful, program:

$$\underset{u}{\text{Min}}(Eu_1(y)|A)/2 \tag{1.4}$$

$$\text{s.t.} \begin{cases} u \text{ implements } A & \\ u_2 = mu_1 & \text{(budget balance)} \\ E(u_1(y)|A)/2 \leq u_1(y') & \text{for every } y' \text{ in } Y. \end{cases}$$

It should be noted that 1.3 readily implies that the classical Green and Porter (1984) model of collusion with hidden action does not achieve efficiency. For, if firms are producing monopoly quantity, it is profitable for both of them to increase production. So no budget balanced transfer scheme can enforce monopoly output.

Formulas for finding the sets $I(\delta)$ are given in the appendix, along with proofs of previous assertions.

2. Proofs

2.1. Recursive methods. The proofs use the recursive methods of Abreu et. al. (1990). They show that every equilibria v can be factored in a strategy α in the current period and a continuation reward function $u : Y \longrightarrow \mathbb{R}^2$, such that u δ-implements α:

Definition 2. *A reward function u δ-implements α if*

$$\begin{aligned} v_i &= (1-\delta)g_i(\alpha) + \delta E(u_i(y)|\alpha) \\ &\geq (1-\delta)g_i(a_i,\alpha_{-i}) + \delta E(u_i(y)|a_i,\alpha_{-i}) \end{aligned} \tag{2.1}$$

that is,

$$\begin{aligned} & E(u_i(y)|\alpha) - E(u_i(y)|a_i,\alpha_{-i}) \\ \geq\ & \frac{1-\delta}{\delta}(g_i(a_i,\alpha_{-i}) - g_i(\alpha)) \end{aligned}$$

So our previous definition of implementation is 1/2-implementation, for when $\delta = 1/2$ the player weights present and future equally.

A key element of the recursive approach is the Bellman map T.

Definition 3. *The Bellman map T_δ is defined for compact subsets of $\mathbb{R}^2$ as*

$$T_\delta(W) = \mathrm{co}\{g(\alpha) + Eu : u \text{ takes on values in } W \text{ and } \delta\text{-implements } \alpha\}$$

Definition 4. *A set W in $\mathbb{R}^2$ is self-generating if $W \subset T(W)$.*

The key facts we will use are summarized as the next remark:

Remark 1. *The set $PPE(\delta)$ is compact, and is the largest fixed point of T. All self-generating sets are contained in $PPE(\delta)$. T takes compact sets on compact sets.*

This facts are well known, and may be found on either Fudenberg and Tirole (1991) or Mailath and Samuelson (forthcoming). Equation 2.1 has an interesting geometric interpretation: v is an average of present gains and continuation values $u(y)$. So, for values on a side $\overline{AB}$, only promises in $\overline{AB}$ are made. This simple fact has important consequences for the Bellman map.

We will omit, for a moment, indexes δ. The above fact translates as $T(W) \cap \overline{AB} \subset T(W \cap \overline{AB})$. But since $T(PPE) = PPE$, we have

$$\begin{aligned} PPE \cap \overline{AB} &= T(PPE) \cap \overline{AB} \\ &\subset T(PPE \cap \overline{AB}) \end{aligned} \tag{2.2}$$

Which results in the following remark:

Remark 2. *The closed interval $I(\delta)$ is the largest self-generating closed interval in $\overline{AB}$.*

Proof. $I(\delta)$ is a closed interval, for it is the intersection of $\overline{AB}$ with PPE, which is know to be closed and convex. By self-generation every self generating interval in $\overline{AB}$ is contained in $I(\delta)$. And, by 2.2, $I(\delta)$ is self-generating. □

2.2. The equilibria for general $\delta < 1$. By remark 2, $I(\delta)$ is the largest self-generating interval in $\overline{AB}$. But the restrictions for an interval to be self-generating are linear. So, the largest one, $I(\delta)$ may be defined by a linear program. We note this observation as the following lemma:

Lemma 2. *Let $a(\delta)$ be a point in $\overline{AB}$ with $a_1(\delta)$ equal to*

$$\min_{u,w} a_1 \tag{2.3}$$

$$s.t. \begin{cases} a = (1-\delta)A + \delta u \\ b = (1-\delta)A + \delta w \\ u\ \delta\text{-implements } A \\ w\ \delta\text{-implements } B \\ u \in \overline{AB} \\ w \in \overline{AB} \\ a_1 \le u_1(y) \le b_1 & \text{for every } y \text{ in } Y. \\ a_1 \le w_1(y) \le b_1 & \text{for every } y \text{ in } Y. \end{cases}$$

and $b(\delta)$ defined by $b_1(\delta) = \mathrm{Max}_{u,w}\, b_1$.with the same restrictions. If $a_1(\delta) \le b_1(\delta)$ then $I(\delta) = [a(\delta), b(\delta)]$. Otherwise $I(\delta)$ are the static equilibria in $\overline{AB}$.

2.3. Proof of lemma 1. Consider, for a moment, the left half of program 2.3:

$$\min_{u} a_1 \quad \text{s.t.} \begin{cases} a = (1-\delta)A + \delta u \\ u \ \delta\text{-implements } A \\ u \text{ belongs to the } \mathit{line}\ \overline{AB} \\ a_1 \le u_1(y) & \text{for every } y \text{ in } Y. \end{cases} \tag{2.4}$$

the key observation is that its solution does not depend on δ.

Proof. Say a and u' are feasible for a discount factor δ'. Then it is trivial to verify that a and

$$u = \frac{\delta - \delta'}{\delta(1-\delta')}a + \frac{1-\delta}{\delta}\frac{\delta'}{1-\delta'}u'$$

are feasible for δ. □

Moreover, as δ approaches 1, u approaches a. So the ignored constraints are, indeed satisfied if $a_1 < b_1$.

Now, consider program 2.4 with $\delta = 1/2$, and the variable $\widetilde{u} = u - A$ substituting u. Then program 1.4 obtains. The equivalence between programs 1.3 and 1.4 can be checked directly.

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