W. E. Armstrong on Intransitivity of Indifference and his Influence in Choice Theory

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Abstract

In this paper we reconstruct the trajectory of a branch of choice theory in that the relation of indifference is intransitive. This trajectory begins with the work of W.E. Armstrong, a British economist that worked in this theme from 1939 to 1958. In 1956 the mathematician R.D. Luce formalized the Armstrong's idea and this formalization influenced others authors in several areas like economics, psychology, operational research, and philosophy, among others. In this reconstruction we also analyzed and synthesized the data of published papers with references to Armstrong's papers and Luce paper.

Keywords: Intransitivity; W.E. Armstrong; R.D. Luce; Semiorder.

JEL codes: B21, B16, B31.

Resumo:
Neste artigo nós reconstruímos a trajetória de um ramo da teoria da escolha na qual a relação de indiferença é intransitiva. Esta trajetória começa com o trabalho de W.E. Armstrong, um economista britânico que trabalhou neste tema entre 1939 e 1958. Em 1956 o matemático R.D. Luce formalizou a ideia de Armstrong e esta formalização influenciou outros autores em várias areas como economia, psicologia, pesquisa operacional e filosofia, entre outras. Nesta reconstrução nós também analisamos e sintetizamos os dados sobre artigos publicados com referencias aos artigos de Armstrong e de Luce.

Palavras chave: Intransitividade; W.E. Armstrong; R.D.Luce; Semiorder.

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1 Introduction

Wallace Edwin Armstrong was a Cambridge anthropologist in the 1920s that moved completely to Economic Theory and that published a series of papers on consumer choice theory from 1939 to 1958. In these papers Armstrong established a strongly critical perspective about the ordinalist revolution of Allen, Hicks, Samuelson, Debreu and others incontestable skillful microeconomists. Armstrong not only made several critics to these dominant branch of the theory as also proposed a cardinal utility function to be employed in choices with both certainty and uncertainty, and a theory of social choice. In all of this endeavor the main idea behind his theoretical construction was the defense that the indifference relation is intransitive.

Each one of us that was trained in microeconomics have learned that the requisites of a rational economic choice are that preferences, normally expressed by the binary relation $\succsim$ ("not worse than"), are both complete and transitive. Therefore, both the relations of strict preference and of indifference may be transitive. The Armstrong's refused in accept the transitivity to indifference relation was based in the notion of "just noticeable difference" that he imported from the experimental psychology. If the agent of choice is not able to differ two almost equal choices, then these choices need be indifferent to her/him. But, the accumulation of small differences can produce a perceptible difference, and then the intransitivity of indifference. This simple idea is out of mainstream economics and Armstrong is not an often remembered reference in economics. In addition to this, Armstrong did not worked with a rigorous mathematical approach like that of the authors of ordinalist revolution or even the cardinal proposal of von Neumamm and Morgenstern to choices under uncertainty.

Notwithstanding, Armstrong's ideas become part of the influences of mathematicians that worked with economics, psychology and related areas. R. D. Luce formalized the notion of a *semiorder*, a form of ordination that support the transitivity of preference relation but not of indifference relation in an utility representation. P. C. Fishburn made several extensions of both Armstrong's and Luce's ideas. This formal branch of choice theory is complementary with the canonical economic theory and also is part of the construction of a formal economic psychology. Our main objective in this paper is reconstruct this trajectory and also employ some bibliographical analysis in order to understand the impact of Armstrong and Luce original publications in academic research.

The Armstrong's papers from 1939 to 1958 have been cited in some papers in the history of economic thought [14, 16, 37, 42, 43, 50, 57]. However, in most of these references the name of Armstrong appears just as a marginal references. The work that spend more attention to Armstrong is [50], but also without so much importance. The paper [50] is a investigation of the work of the mathematician and economist Georgescu-Roegen, and as we will see there was an interaction between this author and Armstrong. There is also a entry in the *New Palgrave* dictionary of economics dedicated to Armstrong [36] (in the 1987 edition, removed in subsequent edition). Although there are relevant information in this entry (we used a lot of it in this paper), the discussion of Armstrong's ideas about utility theory and the intransitivity of indifference are not the main scope of this entry. Therefore, we consider that our work also have some original proposal in discuss Armstrong's ideas about these themes in a main place.

In addition to this introduction and the concluding remarks at the end, this paper has four more sections. In the next section we will present some biographical presentation of Armstrong and his background in psychology, after that we will reconstruct the main ideas in the Armstrong's papers on choice theory, with emphasis on the intransitivity of indifference. Section four presents the formalization by Luce, and in the subsequent section we will analyze and synthesize the data about the published papers that have cites Armstrong's papers and/or Luce paper on semiorder.

2 Biographical notes and some evidence about Armstrong's background in psychology

In our best know all biographical writings about Wallace Edwin Armstrong are in [61, 36, 35, 4] and an entry in the encyclopedia *Who's who* that we didn't have access but that were one of bibliographical basis to [61].[1] Both [61] and [36] informs that Armstrong was born in 1982 while [35, 4] and the *Who's who* entry about Armstrong appoints 1896 as the born year, the order in references suggests that 1896 is the correct year. Without divergences all references informs that Armstrong died in 1980, both events in England.

Following these references we know that Armstrong graduated at Cambridge in moral sciences in 1918 after returned from I World War where served and was wounded and lost a leg in 1915. Armstrong started his academic career as an anthropologist, first in a field research in Papua New Guinea and after teaching at Cambridge, but moved completely to economics in 1930's and his main contribution in this area will be reviewed in the next section. The most complete reference to Armstrong's initial period as anthropologist is in Urry 1985 [61], whom exhibit a comprehensive scenario about Armstrong's formation, his work in Rossel Island (Papua), and, mostly, his years (1922 - 1925) as the first one to teach *social anthropology* at Cambridge with an official post, a precarious position that nearly become stable. Although [36, p. 115] informs that "Armstrong's place in the history of anthropological thought is more secure" as compared with his place in economics, [61] has demonstrated that Armstrong's place in anthropology was also neglected, mainly because his pioneering position in social anthropology is often forgotten. Armstrong's transition to economics started yet in 1920s with supervisions and eventual lectures in Cambridge between 1926 and 1939, from 1939 to 1961 he taught at the University of Southampton, where retired as Professor of Economic Theory [61, p. 414].

Following most of these bibliographers we may affirm that Armstrong as economist had no salient trace of the anthropologist. Gregory and Urry 1987 [36, p.115] notes that Armstrong abandoned all aspects of empirical research and concentrates in "pure theory" in economics. These authors (same page) cite the Armstrong's book *Saving and Investment* [6] as example of this rupture with anthropology and the uncritical use of "Robinson Crusoe" metaphor of economy in this book as a main evidence in that argument.[2] Notwithstanding, about this book, the obituary in *The Times* [4] make a suggestion that the Armstrong's research of the monetary system in Russel Island[3] led to his research about the modern saving and investment system. Although this configures a plausible possibility it is far from our objectives here. The principal aspect that we can observe in Armstrong's economic work is the development of a theory of choice in a close proximity with British economic tradition of Jevons, Marshall, and Pigou. Anyway, in reference to Armstrong's last years at University of Southampton [35, p.1] asserts yet that "many close colleagues knew nothing of his previous career as an anthropologist[...]." Then, it is not wrong conclude that there was a complete reconstructions of Armstrong academic career when he wrote his papers on choice theory from 1939 to 1958.

On the other hand this conclusion doesn't means that Armstrong's academic background had none influence in his research in economics. Following [36, p. 115] there is two main aspects that remained. The first aspect is the Armstrong's reasoning through *a priori* hypothesis, an intellectual choice that [61, p. 428] considers that "may have been a product of Cambridge philosophy" and that certainly facilitated the transition to economics and the work in utility theory.

[1] The another main reference is [4], see the explanation in [61, p. 429, footnote 2].

[2] [35] inform us that Armstrong's book "was overshadowed by Keynes's classic work published in the same year" and certainly it is true, but the book was not ignored. Abba Lerner wrote a complimentary review in *The Economic Journal* [44] and the book was quoted by Hayek [38].

[3] See details and references in [61].

The second aspect, the most important for our investigation, is Armstrong's training in psychology. Following [61, p. 413] Armstrong concentrated in psychology in his last year as a Cambridge student (1918) and then was convinced by W. H. R. Rivers to work with anthropology. Economists (as us before this investigation) maybe don't know nothing about this but there was a strong relationship between experimental psychology and anthropology in that years.[4] The own Rivers is a personification of the intersection of these two research areas. He studied medicine and work as a physiologist and with mental diseases before anthropology, and even after some of his anthropological expeditions (the most famous the Torres Straits Cambridge expedition) he developed psychophysical experiments with his colleague at the National Hospital in Queens Square, Henry Head, between 1903 and 1907 [15].[5] Rivers taught anthropology at Cambridge but without an official position. When he died in 1922 Armstrong assumed his post, but now with an official, although precarious, status that was renewed until 1925 [61, p. 414].[6]

There are evidences that confirms that psychology was not just an academic curiosity to the young Armstrong as an undergraduate student at Cambridge. [61, p.426] classifies that Armstrong's *social anthropology* had a "psychological base" and, in addition to this, informs us that Armstrong was a member of British Psychological Society from 1925 to 1931 [61, p. 418], the same period of his move to economics. As we will see in the next section one of the basis of Armstrong's cardinal utility theory is a psychophysical argument about the human incapacity to distinguish between small differences. Indeed, Armstrong explicitly works with the concept of "just noticeable difference" from psychophysics and also mention the work of psychologists, even if just in general terms. This is a different perspective in comparison with economists that just mention "psychological motives" in loosely terms. Unfortunately Armstrong didn't referenced his theoretical basis of his psychological or psychophysics arguments.

3 Armstrong on choice theory - the papers from 1939 to 1958

We will name the following set of papers [7, 8, 9, 10, 11, 12, 13] the Armstrong's papers (*AP*). In all these papers there is the presence of the theoretical importance of the intransitivity of indifference in choice theory.[7] These set of papers is a *tour de force* in which Armstrong established several critics to the kind of choice theory related with the ordinalist revolution, the branch of theory that he identified mainly with the work of Hicks and Allen 1934 [41], but also with others.[8] Although the own Hicks had considered Armstrong's critics not a rejection [40, p.7, footnote 1.], and another author in 1950s had informed that the ordinalist notion of indifference was related with compensation, while the Armstrong's notion was one about approximation, and that these two "are not necessarily incompatible" [47, p. 30], Armstrong also was in disagree with both the construction of a theory concerned just to choices under certainty [8, p. 1-2.], like that of [41], and the theory of choice under uncertainty present in von Neumann and Morgenstern 1944 seminal work [49], see [12, p. 171, footnote 1] and [13, p. 174,

[4] See for example [60].

[5] Certainly that this presentation of an academic like W. H. R. Rivers is very abridge, then we refer the curious reader to the references in [61] and [15].

[6] [61, p.414] informs us that Armstrong was in a list with others two colleagues to stable posts, but that just one was established. In the same page this author inform us that after this rejection "Armstrong gave instruction to Tripos students in the Ethnology of the Special Area (Melanesia) in 1926-27 and offered lectures and supervision until 1931 [...] but his career in anthropology had effectively ended."

[7] In order to be precise, it is true that in [12] the intransitivity of indifference appears just indirectly, but in coherent terms with precedent Armstrong's papers.

[8] In addition to [41] Armstrong cited first [3] and [39] in the footnote that is attached to title his 1939's paper, but also a set of papers by prominent authors - include O. Lange and P. A. Samuelson - of the 1930s as the main literature for his topic.

footnote 1].

In *AP* the author suggested cardinal utility functions for consumer theory with and without uncertainty, and also a theory of social choice and welfare, clearly an very ambitious project.[9] This cardinality is based in the idea of the measurement and comparison of the differences or distances of the utilities of two possible choices. As well as all measurement process, the utility comparison will be always imperfect and this would be the root of intransitivity of indifference for Armstrong.

In this section we will do an intensive use of direct quotes from *AP*, mainly from [7]. Our objective is conduce a close presentation of Armstrong's ideas, arguments, and rhetorical construction against the ordinalist revolution, but with the focus on the theme of intransitivity of indifference relation and Armstrong's use of psychophysical arguments. We will also try to observe Armstrong's theoretical perspectives, but without an unnecessary speculative view.

The year of Armstrong's first publication in this endeavor (1939) is just a few years (or less than this) after of the main works of Hicks, Allen, and Samuelson in the theme of utility representation. Nor had already in microeconomic theory the formal demonstration of the topological requisites for this representation - introduced by Debreu in 1952 [17, cap. 6] - that generalized this form of representation. Although microeconomics was becoming more and more mathematical in form, there was yet room for old style microeconomists and also certainly was not clear for all practitioners which would be the dominant mathematical strategy in this area. We can affirm that Armstrong was in the old style set of theorists but that he was also able to interpret and discuss with the set of more mathematical microeconomists by the use of simple and intuitive mathematical representations.

Today we all know the process of unification of neoclassical microeconomic theory, but at 1939 stage Armstrong could not be accused of anachronism when he identified not one but three groups of microeconomists with (maybe almost) the same degree of importance: the *Lausane School* with the radical idea of utility based only in preference relations, the *introspectionists* and the *behaviorists*. Differently of the first group, microeconomists from the last two groups could also be divided between those who employ determinate (cardinal) utility functions or indeterminate (ordinal) utility functions. Armstrong clearly was from the subset of introspectionists that employ cardinal utility functions, while Hicks and Allen [41] - in Armstrong's view - were close to behaviorists in ordinal utility functions subset.

In Armstrong's perspective anyone that did not employ an cardinal utility function will eventually deduce that when the agent of economic choice selects a set of alternatives that are indifferent to her/him - an *indifference class* - the relation of indifference between the members of this class will necessarily be transitive. To Armstrong this was a clear evidence of the inconsistency of these branch of choice theory because the intransitivity of indifference relation would be an well established fact. In his own words:

> But it is a well-known fact that it is possible to be indifferent as between two alternatives A and B and as between B and C, while there is preference for A over C, *i.e.*, the relation of indifference is not transitive. [7, p. 457.]

Observe in this passage that Armstrong did not reclaim for himself the originality of the idea about intransitivity of indifference. Instead of, he employs this result as an "well-know fact" in his strategy to expose the inconsistency of ordinal representations. But Armstrong did not informs us any reference or economic tradition to this "well-know fact". At this point, all that Armstrong offered is the follow example (his only economic or choice example in all *AP*):

[9] Although our focus here is the tradition in intransitivity of indifference, Armstrong's influences in welfare theory and his investigation of choice under uncertainty could also render a future investigation.

> If, for example, we consider a series of alternatives consisting of various combinations of bread and cheese, obtained by the substitution of bread for cheese, it is possible, starting with one alternative, to make a progressive substitution of bread for cheese which preserves our indifference to the substitution as between successive steps and yet to arrive at a combination of bread and cheese which is preferred or conversely preferred to the original combination. [7, p. 457, footnote 1.]

One reference that Armstrong could have cited is Georgescu-Roegen 1936 [31, pp. 568-575]. This author worked with the concept of *psychological threshold* in order to deal with psychophysical limitations of the agent of choice. In his psychophysical example, Armstrong will employ the term *just perceptible difference* [7, p. 465]. Differently of Armstrong, [31, p. 569, footnote 5.] informs us a reference, a book of G.S Fullerton and J.M. Cattel (possibly [29], the reference in [31] is incomplete), and also works in probabilistic terms, differently of Armstrong. Some years later, Georgescu-Roegen 1954 [32, p. 523, footnote 5.] would inform us that:

> The fact that the psychological threshold destroys transitivity of indifference is the main point of Armstrong's paper (1); very likely, he must have ignored the author's paper, to argue further that the threshold implies cardinal utility.

In [32] "Armstrong's paper (1)" is [7] and "the author's paper" is [31]. The last one of the *AP*, [13], is a response to [32] and from the historical perspective it is important register the Armstrong's answer

> Mr. Roegen drew attention to the non-transitivity of indifference as early as 1936 in that brilliant article [...]. I was unaware of this article when I laid so much stress on this non-transitivity in 1939 and must apologise for the omission of any reference. [13, p. 174, footnote 3].

In our best know the first one that had related the work of these two authors in these theoretical context was Rothenberg 1953 [53, p. 249.], in a paper critical of Armstrong ideas on welfare theory [10] and one that received a replay by Armstrong in the same year [11]. Therefore, we can conclude that Armstrong was conscious of Georgescu-Roegen's 1936 paper and his contribution to the microfundamentation in psychological basis of the phenomena of intransitivity of indifference in economic theory at least by 1953. It is not an accusation about the veracity of the Armstrong apologize to Georgescu-Roegen, it is just an attempt to better reconstruct this episode in economic thought. Therefore, we conclude that in Armstrong's view the intransitivity of indifference really was a known fact, one that possibly was discussed in economic classes and one economic fact that he felt that did not deserve an specific reference.

Back to the argumentation in [7] we can observe the first mathematical expression of his cardinal interpretation of the intransitivity of indifference:

> It follows, at once, that the relation of indifference is not transitive, for if, for example, the difference between U_1 and U_2 is $+a$ and between U_2 and U_3 is $+b$, then the difference between U_1 and U_3 is $+(a+b)$. If a difference of utility of $< c$ gives rise to indifference, while a difference of $\not< c$ gives rise to preference, then if $a < c$ and $b < c$ but $(a+b) > c$ then the alternative of utility U_1 will be preferred to the alternative of utility U_3, even though there will be indifference to the alternatives of utilities U_1 and U_2 and to the alternatives of utilities U_2 and U_3. [7, p. 461.].

The first impression about this cardinal utility representation can induces us to interpret that Armstrong really works with the idea of a direct measure for utility. However, this author had a good notion of the impact of this idea at that stage of microeconomic thought and worked his arguments in order to dissipate this impression:

> [Utility-theorists] have been accustomed to assume a quantitativeness of utility because of a supposed quantitativeness of an observed quality that they called utility. This, however, is not the critical consideration, and it may well be the case that there is no observable quality, called utility, which can be observed as in any sense quantitative. What *can* be observed by the introspectionist, however, is something that can be called degrees of preference. I can clearly observe by introspection the fact that I prefer A to B more strongly than I prefer A to C, and I can think of a preference diminishing to a point at which I would assert that the preference has passed into indifference. [7, p. 462. In brackets inserted by us.].

With this concept of *degrees of preference* Armstrong could define the idea of a measure of the distance, in terms of utility, between two choices without none requisite of a direct measurement of utility, but just with the perspective of introspectionism. Armstrong also explained that these judgments (measures) in an introspectionist interpretation is always an exercise of estimate and an imperfect approximation:

> A is preferred to B because it is estimated that the pleasure attaching to A if A were chosen would be greater than the pleasure attaching to B if B were chosen. If there is indifference between A and B it is estimated not that the pleasure attaching to each would be precisely equal, but simply that neither can be regarded as greater (so far as our necessarily approximate estimate suggests) than the other. Now measurement always implies approximation. [7, p. 463].

Therefore, in the appraisal of the subset of indifferent choices this degree of what we can call now bounded rationality gains a bigger importance and the impression is that Armstrong works with the notion of a self awareness of the agent about her/his own limitations. Following the argumentation summarized in [52], the dominant branch of microeconomic choice theory established the economic agent not as an human prototype but just as the unit of rational economic choice, an analytical instrument. It was never an universally accepted methodological strategy and the question about how (rational) humans make economic choices is even present at the core of mainstream economics in different perspectives, being the (actual) behavioral economics the most prominent example. Armstrong certainly adopted an enlarged or more "realistic" ambient for his microeconomic theory.

At the same time Armstrong was a radical cardinalist, one that had differentiated economists that accepts "cardinal preference" (a softer cardinalism) from those who accepts "cardinal utility" [12, p. 176.]. Certainly that Armstrong would include himself in the second group. But, possibly for a considerable share of actual microeconomists (we think that for a majority) cardinalism does not look like more realistic than ordinalims. The successful strategy employed by Armstrong was link cardinality with psychophysical arguments, and this led his theory to survival. We will turn us to this perspective now.

As we have observed Armstrong related the intransitivity of indifference to the inherent imperfection of a measurement. However, there must be differences between physical measurements like the measure of the distances between two bodies, a process that involves separately the agent of measure and an instrument of measurement (*e.g.* a measuring tape), and the economic choice by an agent who is at the same time the agent of choice and the own source of the subjective tastes, preferences, or internal

knowledge and appreciation about the different options that will be compared or measured. However, it is important to observe that Armstrong did not take the economic choice just as only an analogy of the measurement process and, in order to offers a most plausible approximation, he had appealed to a psychophysical example.

Interestingly, Armstrong choose the perception of qualities of colors as brightness and hue in his psychophysical examples. This topic was of large interest for social or cultural anthropologists, but there is none reference to the Armstrong's first academic area in *AP*, and in special with respect to this topic. Although colors are physical phenomena and in our days subject to classification and verification by computers and scanners (external instruments of measurement), Armstrong [7, p. 464] refereed to "[...] an introspected quality such as colour" in order to make a comparison with the (possibly) introspected variable called utility. He [7, p. 465] informed us that qualities of colors like brightness or hue can be used in order to orderly different colors in measurable intervals:

> All that this means, of course, is that assuming hue to be a simple characteristic of colour, then if we vary hue away from *A* through *B* to *C*, *B* can be said to have a measurable position between *A* and *C* in terms of the relation between *A* and *C*, or, in other words, there is a way of measuring position in a series of hues in terms of any two positions taken as standard. [7, p. 465]

Then, also in this kind of measurement there is just the possibility of an imperfect measure. In addition to this, it is in this example that Armstrong works for the first time with the notion of "just perceptible difference":

> But if I am unable to detect a difference of brightness between *A* and *B* there is no such implication of identify of brightness and I should be surprised if they really were identical. It is for this reason that such measuring-rods as "just perceptible difference of brightness" or "apparent equality of brightness" are necessarily regarded as very imperfect measures[.] [7, p. 465-466]

As we have insisted in this paper Armstrong did not offer any specific references for these ideas, but almost one generic reference to psychologists appears later in [11]:

> Psychologists have repeatedly emphasized the point in explaining the non-transitiveness of the relation 'not perceptibly different' in judgements about the constituent parts of fields of perception. [11, p. 268]

As we will see in the next sections R. D. Luce 1956 [45] developed the notion of *semiorder*, a formal structure of binary relations that obeys Armstrong's notions about the intransitivity of indifference. More than this, Luce was a mathematician that works close to psychologists and had strong impact in this area and also in economics, among others research areas.[10] But, one of first appreciations of Armstrong's arguments by a psychologist was not favorable. In his influential paper Edwards 1954 [18], making reference to [7, 8, 9], informs us that

> Incidentally, failure on the part of an economist to understand that a just noticeable difference (j.n.d.) is a statistical concept has led him to argue that the indifference relation is intransitive[.] [18, p. 388.]

[10]It is not a disputable fact that Luce was mainly concerned with mathematical psychology, see [30] for a presentation of the main Luce works in this area. What we will argue in the next section is that Luce was more close to the economic perspective at the beginning of his career.

There is none reference about this critics in the last of *AP* [12, 13] that were published after [18]. After that, Luce and Edwards making an important contribution to the mathematical construction behind the notion of "just noticeable difference" in [46],[11] but in this paper there is none mention to *AP* or [45]. The analyses of the importance of Armstrong's ideas and also the importance of semiorder in the work of Luce after [45] needed be studied in a separate paper. Here we will just observe the main arguments of [45] (next section) and the impact of *AP* and [45] in terms of references (in the last section).

4 The formalization of a semiorder by R.D. Luce

[30, p. 493] inform us that Robert Duncan Luce was born in 1925 and that had obtained his PhD in mathematics in 1950 at MIT. In Luce's 1956 paper [45, p. 178, footnote 1] it is informed that "This work was undertaken and completed when the author was a 1954-55 Fellow at the Center of Advanced Study in the Behavioral Sciences, Stanford, California." This brief introduction to Luce and to his 1956 paper informs us that not only Luce had a brilliant formation and that the beginning of his academic career was at one of the most prestigious centers before the age of 30, but also help us understanding that both psychophysics and Armstrong's ideas was sufficiently relevant topics for a highly promising academic at the middle of 1950s, when Armstrong was yet an active researcher.

However, before we developed the use and relation of Armstrong's ideas in [45] we will present some evidence that Luce at this stage of his career was closest to the economic perspective than others authors that works with the theme of intransitivity in binary relations like May [48] and Tversky [59]. Both May and Luce published at *Econometrica* at the first half of 1950s and employed some reflections and references from psychology in order to contest in variable degrees the axioms of transitivity in the canonical economic choice theory. However, although May [48] had emphasized the intransitivity for preference relation and circular behavior, Luce [45] concluded that the axiom of transitivity of preference relation should be kept:

> It is necessary if one is to have numerical order preserving utility function, and such functions seem indispensable for theories - such as game theory - which rest on preference ordering. Furthermore, there is the important subjective contention that a "rational" preference ordering *should* satisfy the transitivity condition. For example, Savage [...] writes "...when it is explicitly brought to my attention that I have shown a preference for f as compared with g, for g as compared with h, and for h as compared with f, I feel uncomfortable in much the same way that I do when it is brought to my attention that some of my beliefs are logically contradictory." Our quarrel with the axioms of utility theory does not lie here, and we too shall take the attitude that, at least for a normative theory, the preference relation should be transitive. [45, p. 178-179] [12]

When Luce rejects the possibility of an intransitive relation of preference and the consequent possibility of circular choices - as observed in empirical experiments - he employs an strategy that we consider typical of economic rhetoric, he employs both an instrumental argument - the necessity of utility functions in some uses - and a normative (and in some degree introspective) argument supported by a prominent author like Savage.

[11] In addition to this [30, p. 495] inform us that [46] was the work that employed for the first time the solution of a functional equation in both psychology and economics.

[12] The reference of the quote of Savage that we covered with brackets can be found in [55, p. 21].

On the other hand, with respect to the transitivity of indifference relation Luce [45, p. 179] considers "that there is little defense for supposing" its validity. In the sequence [45, p. 179] he inform us that: "The author who has most repeatedly questioned this assumption is Armstrong[.]". In this passage Luce cited [7, 8, 9, 10], inclusive with a quote of [9].[13] Luce [45, p. 179] also remits, explicitly, to psychophysics as a source for "empirical evidence" for the intransitivity of indifference - what contradicts [18] - and offers an example that became classic in this discussion:

> Find a subject who prefers a cup of coffee with one cube of sugar to one with five cubes (this should not be difficult). Now prepare 401 cups of coffee with $\left(1+\frac{i}{100}\right)x$ grams of sugar, $i = 0, 1, \ldots, 400$, where x is the weight of one cube of sugar. It is evident that he will be indifferent between cup i and cup $i+1$, for any i, but by choice he is not indifferent between $i = 0$ and $i = 400$. [45, p. 179]

Notwithstanding, Luce was conscious of the statistical nature of psychophysics and concluded:

> The theory we shall obtain yields a non-statistical analogue of the "just noticeable difference" concept of psychophysics. [45, p. 180]

In this paper Luce informs about psychophysics but does not named any author. An economist that works with Luce's semioerder notion is Gilboa [33, p. 65-71.]. In this passage of Gilboa's book there is a succinct presentation of Weber's law and its relation with the notion of just noticeable difference.[14] We believe that these bibliographical evidences are sufficient to link definitely Armstrong's vague references to an specific tradition in experimental psychology. We will turns now to the Luce's definition of semiorder:

> Let *S* be a set and *P* and *I* be two binary relations defined over *S*. *(P,I)* is a *semiordering* of *S* if for every *a*, *b*, *c*, and *d* in *S* the following axioms hold:
>
> S1. exactly one of *aPb*, *bPa*, or *aIb* obtains,
> S2. *aIa*.
> S3. *aPb*, *bIc*, *cPd* imply *aPd*,
> S4. *aPb*, *bPc*, *bId* imply not both *aId* and *cId*. [45, p. 181]

Fishburn and a coauthor [28, p. 170] are explicit that the concept of semiorder was already presented in Armstrong's consumer theory. As we will see in the next section Fishburn is the author that have the stronger importance in the diffusion of both Armstrong and Luce works by the publication of several papers (among others forms of academic works) with references to these authors.

5 The presence of Armstrong's papers and Luce (1956) in published papers

In order to understanding, and in some part measure, the impact of Armstrong's papers and Luce 1956 [45] we organized the references to these works in papers published in academic journals by year,

[13] This quote is the passage that close [9, p. 122]: "The non-transitiveness of indifference must be recognized and explained on any theory of choice, and the only explanation that seems to work is based on the imperfect powers of discrimination of the human mind whereby inequalities become recognizable only when of sufficient magnitude."

[14] The Weber's law was developed by the physician E. H. Weber in 1834, see the complete reference in [33, p. 65 and p. 198].

research areas, most frequent journals and authors. We excluded others forms of academic publications (books, working papers, chapters, annals of congress and so on) for pragmatic reasons, there is more uniformity in the set "papers published in academic journals" than in the another forms. But, also, there is the fact that published papers are a more robust type of publication, in terms of the acceptability by the pars. In the last part of this section we will also employ some statistical procedure in order to better understand the relation of these two sets of papers.

We will begin with the analyses of references to *AP*. With the exception of [7] there are disposable data for all these papers in *Web of Science* (*WoS*),[15] then we choose this database as us preferential source and we have complemented with *Google Scholar* (*GS*) database as a source for references to [7], but also with references in these papers.[16] Thus, we construct and verified a database of references to *AP* in a conservative way. We also exclude the own Armstrong's papers in which he made reference to himself.

There are 122 papers in this set, and they were published between 1948 and 2019. Figure 1, left graph, shows citations by year (columns) and the moving average of 10 years (line). The moving average behavior suggests that there was a positive trend between 1960s and 1970s and a (close to) stationary behavior in 1990s and after. Figure 1, right graph, shows the accumulated number of papers in this set (points) and the linear adjustment for this series. The angular coefficient of this linear regression is 1.68 (p-value $= 2\text{x}10^{-16}$ and R^2 adjusted equals 0.9896). Obviously that a growth series would have a highly significant positive angular coefficient, we informed the stats just in order to be precise, but the visual adjustment suggests that our hypothesis of a quasi-stationary behavior for the moving average is a good approximation, and the angular coefficient informs that there was close to 0.7 new citations by year (in average) of *AP*. The years with more citations are 1970 and 2018, each with 7 papers.

In table 1 we organized the citations by decade and classified them in three areas of research: economics, psychology, and "other areas" (*OA*). This classification is based in the scope of the journal, and has an certain hierarchical order from economics to psychology to other areas in this procedure. For example, *Journal of Mathematical Economics* and *Economics & Philosophy* were both classified in economics group and not in *OA*, although there is journals of both applied math and of philosophy in the *OA* group. In the same sense *Journal of Mathematical Psychology* was classified in psychology group. We applied the same classification criteria for references to Luce 1956 [45] and in this set there is one reference in *Journal of Economic Psychology*, this one was classified in economics group and not in psychology group. On the other hand journals with frequent publications of papers highly related with economics but that are multidisciplinary journals, like *Theory and Decision*, were classified in *OA* group. The main evidences in table 1 are about the share of publications by area and some evidences about the irregular distributions by decades that provides additional evidences to our graphical analysis. Close to a quarter (23.8%) of the papers were published in 1970s, and the decades with less publications are 1960s and 1980s, after that the publications become regular and close to 20 by decade. This is in line with the behavior of the moving average. The news here are about the division in research areas. Exactly a half of the papers are in economics, followed by *OA* (36.1%), and just a residual part (13.9%) in psychology.

The 122 papers with references to *AP* published between 1948 and 2019 are in 66 different journals, but 54 of these papers (44.3%) were published in just 10 different journals, and about one third (36.1%) were published in just 6 journals. The information about these most frequent journals are in table 2. In this table we can observe that *Journal of Mathematical Psychology* (*JMP*) is a very privileged

[15] We have used the software *Bibliometrix* for *R* [5] in part of our analysis.

[16] *Jstor's* database was also considered as our source, but as we will see the references are in so many diverse research areas that we would need consider the search in virtually all *Jstor's* database and this would be prohibitive and just with an uncertain marginal gain, if any.

publication in this set with 14 papers between 1970 and 2019. Moreover, among the 17 papers in journals of psychology that have cited Armstrong, 14 were published in *JMP*.[17] We will see that this journal is the most frequent also in references to Luce 1956 [45]. We can also observe in table 2 that *Econometrica* was the privileged journal before *JMP*, with 9 papers between 1953 and 1973. Certainly it is not without significance this transition from a highly prestigious journal in economic theory to an also prestigious journal, but out of the strict domains of economic thought. Table 2 also shows others highly prestigious journals and predominantly in economics but also in *OA*. *Mathematical Social Sciences* has evident scope, while *Synthese* is a journal in epistemology, methodology and philosophy with several intersections with economics.

The authors that most frequently have cited *AP* (excluded himself) is P.C. Fishburn (16 papers, from 1968 to 1997), F.S. Roberts (5 papers, from 1970 to 1985), and the co-authors J. C. Candeal and E. Indurain (4 papers, from 2002 to 2011). As we have yet discussed P. C. Fishburn is one of the principal actors in this tradition. Only in the year of 1970 Fishburn published 6 papers with reference to *AP* [21, 24, 23, 25, 20, 26], two of these in *Journal of Economic Theory* and one in *Econometrica*.

Last, but no least, the set of most cited papers (*WoS* data) with reference to *AP* are formed by: Allais 1953 [2] (1575 citations), Edwards 1954 [18] (1178), Zadeh 1971 [62] (1450), Luce 1956 [45] (502), Sen 1971 [58] (335), and Fishburn 1974 [27] (286). We have already discussed [18] reference and his critics to Armstrong, but not all economists know the importance of L. A. Zadeh. He is the father of "fuzzy logic", the set of conceptual and instrumental tools designed to work with categories that are of no "crisp" classification. There is evident intersections between this theme and intransitivity of indifference by the arguments of Armstrong and Luce. The principal journal of this research ares is *Fuzzy Sets and Systems* and some papers with reference to our authors have been published there. In these privileged set we need, also, recognized the importance of the classical paper of A. Sen [58]. In this paper Sen relates the ideas of both Armstrong and Luce with his notion of quasi-transitivity [58, p. 314]. We include these most frequent cited papers in order to explore some qualitative attributes of Armstrong citations. Then, although *AP* can be considered an obscure reference, because there are only few references to his work, this is in some sense a prejudice, and a set of high prestigious authors from different areas have mention (in favorable or critical perspective) his ideas.

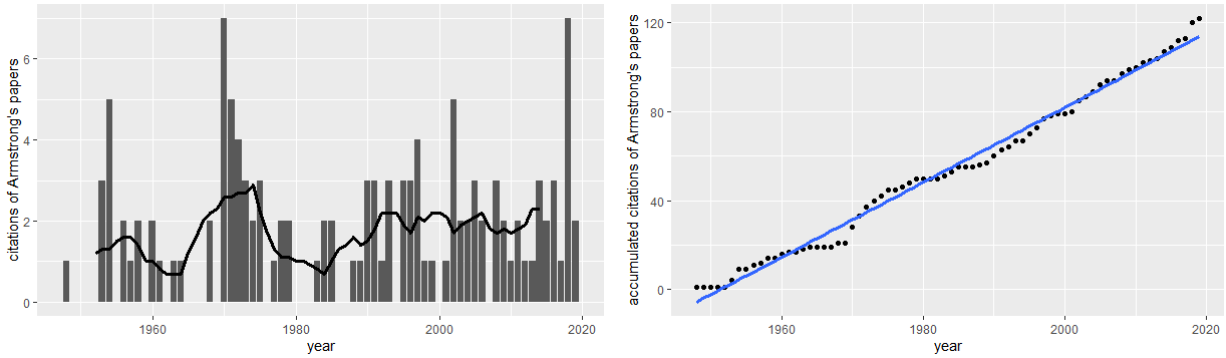

Figure 1: Citations of Armstrong's papers by year and accumulated (1948 - 2019)
Source: Our elaboration, data from *WoS* and *GS*

[17] The three others are [51, 18, 1].

Years	Research Area			Total
	Economics	Psychology	Other Areas	
1948 - 1959	13 (92.9%)	1 (7.1%)	0 (0%)	14 (11.5%)
1960 - 1969	6 (85.7%)	0 (0%)	1 (14.3%)	7 (5.7%)
1970 - 1979	14 (48.3%)	2 (6.9%)	13 (44.8%)	29 (23.8%)
1980 - 1989	2 (28.6%)	2 (28.6%)	3 (42.9%)	7 (5.7%)
1990 - 1999	11 (50%)	4 (18.2%)	7 (31.8%)	22 (18%)
2000 - 2009	4 (20%)	3 (15%)	13 (65%)	20 (16.4%)
2010 - 2019	11 (47.8%)	5 (21.7%)	7 (30.4%)	23 (18.9%)
Total	61 (50%)	17 (13.9%)	44 (36.1%)	122 (100%)

Table 1: Citations of Armstrong's papers for decade and for research area.
Source: Our elaboration, data from *WoS* and *GS*

Journal	Research of Area	Number of Papers	First Year	Last Year
Journal of Mathematical Psychology	Psychology	14	1970	2019
Econometrica	Economics	9	1953	1973
Journal of Economic Theory	Economics	5	1970	1995
Mathematical Social Sciences	Other Areas	5	1995	2005
Synthese	Other Areas	5	1970	2013
Oxford Economic Papers	Economics	4	1953	2005

Table 2: Most frequent journals with references to Armstrong's papers.
Source: Our elaboration, data from *WoS* and *GS*.

Now, we will employ the same analysis to the references about Luce 1956 [45]. *WoS* database for this work computes 413 papers with references to [45] between 1957 and 2019. Figure 3 (left graph) shows the citations by year (columns) and the moving average of ten years (solid line). It is evident that the behavior of the this time series isn't stationary. In addition to this, the moving average growth at different rates in time. These rates are bigger both at the initial and at final years. Figure 3 (right graph) shows the empirical time series for accumulated citations (solid points) and the linear adjustment to this series. The angular coefficient is 6.37 (p-valor $= 2\text{x}10^{-16}$ ans R^2 adjusted $= 0.950$). As we have already informed the stats of a linear regression on a growth series are expected to be highly significant. Therefore, we will appeal to the visual adjustment in order to defend that this linear adjustment is not so good for initial and final data. To the last years the linear behavior underestimate a growing tendency.

The year with more citations is 2018 (21 occurrences) followed by 2015 (20 occurrences). In the table 3 we can observe the total of citations by decade and among categories. The total of citations between 2010 and 2019 is almost the double of that observed between 2000 and 2009, and more than

one third part of total (34.9%). In relation with the principal research area, more than one half of references to [45] are in *OA* group in the entire period (57.6%) and also by decade, with the exception of the period from 1957 to 1969. The remaining of papers are the in economics and psychology groups in near to equal proportions. From the subset of 238 papers initially classified as *OA* we could reclassify 87 in journals of math, 29 in decision making (multidisciplinary), 27 in philosophy, and 20 in operation research (a total of 163 papers). The 413 papers with reference to [45] are in 166 different journals, but 184 of these papers are in just 11 journals, this information is in table 4. Moreover, 68 of these papers were published in *JMP*. All these information revels that (i) references of [45] can be found in an great variety of research areas and (ii) journals concerned with math are of highly frequency in this set. The authors with the larger number of papers is P. C. Fishburn (45 papers, from 1968 to 2001), E. Indurain (20 papers, from 2002 to 2018), and A. Giarlotta (11 papers, from 2013 to 2019). The prevalence and the number of papers by Fishburn is indisputable, but the other two authors and mainly Giarlotta presents a significant number of papers in very recent years. The papers with most citations (*WoS* data) in this set are the already refereed Zadeh 1971 [62], the classical paper of Tversky 1969 [59] (1164 citations), Greco et al 1999 [34] (289), Fishburn 1970 [22] (246), Scott 1964 [56] (244), Fishburn 1982 [19] (233), and the a relatively recent paper of Ariel Rubinstein 2003 [54] (189).

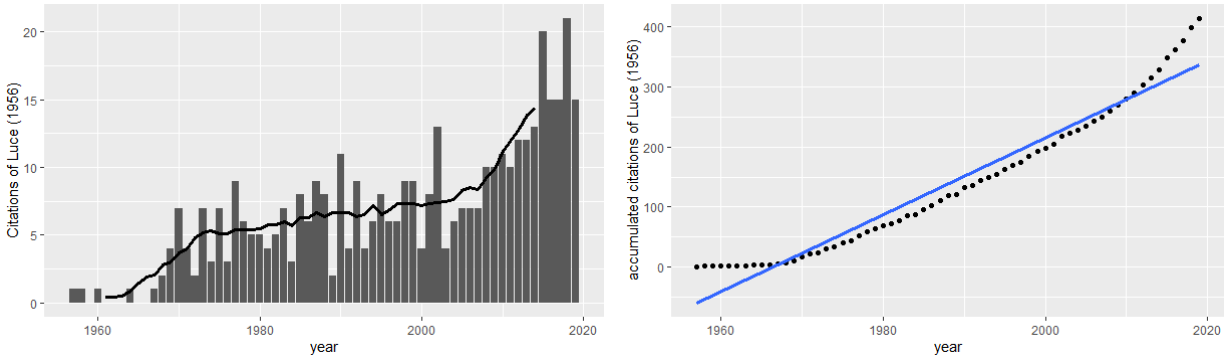

Figure 2: Citations of Luce (1956) by year and accumulated (1957-2019)
Source: Our elaboration, data from *WoS*

Years	Research Area			Total
	Economics	Psychology	Other Areas	
1957 - 1969	2 (18.2%)	5 (45.5%)	4 (36.4%)	11 (2.7%)
1970 - 1979	12 (22.6%)	13 (24.5%)	28 (52.8%)	53 (12.8%)
1980 - 1989	12 (21.1%)	11 (19.3%)	34 (59.6%)	57 (13.8.4%)
1990 - 1999	19 (26.4%)	12 (16.7%)	41 (56.9%)	72 (17.4%)
2000 - 2009	13 (17.1%)	15 (19.7%)	48 (63.2%)	76 (18.4%)
2010 - 2019	28 (19.4%)	33 (22.9%)	83 (57.6%)	144 (34.9%)
Total	86 (20.8%)	89 (21.5%)	238 (57.6%)	413 (100%)

Table 3: Citations of Luce (1956) for decade and for principal research area.
Source: Our elaboration, data from *WoS*.

Journal	Research Area	Number of Papers	Older	Latter
Journal of Mathematical Psychology	Psychology	68	1964	2019
Mathematica Social Sciences	Other Areas	18	1983	2014
Theory and Decision	Other Areas	16	1972	2014
Synthese	Other Areas	12	1970	2013
European Journal of Operational Research	Other Areas	11	1986	2018
Order-A Journal on the Theory of Ordered Sets and its Applications	Other Areas	11	1986	2016
Econometrica	Economics	10	1958	2019
Journal of Economc Theory	Economics	10	1970	2016
Social Choice and Walfare	Other Areas	10	1985	2019
Discrete Mathematics	Other Areas	9	1982	2006
Journal of Mathematical Economics	Economics	9	1983	2016

Table 4: Most frequent journals with references to Luce (1956).
Source: Our elaboration, data from *WoS*

Once we had observed and organized the data about references to *AP* and Luce 1956 [45], we can now work with the two sets jointly. At this point there is no doubt that [45] is largely a more influential reference than *AP*. Here we want observe when the researchers have or not cited *AP* in works that make reference to [45]. In order to establish some empirical measure of this relation we verified in the set of papers with reference to [45] (413 papers) which of these have also cited *AP*, after that we estimate the correlation coefficients between this binary variable and the binary variable for each of our research areas (economics, psychology and *OA*) that classified the paper that cited [45] . The results are in table 5. There are 56 papers with citations of almost one of the *AP* and [45] simultaneously. For the economics group the coefficient of correlation is positive (0.128) and highly

significant (p-value below 1%). For *OA* the coefficient is negative (−0.118%) and also with highly significant (p-value approximately 1.61%). On the other hand, the correlation coefficient in psychology group is statistically equals to zero. The interpretations is that when in economics area the citations of *AP* are to be expected, and when in *OA* these citations are not to be expected. These statistical results points that for the authors who work with the reference to [45], those who publish in economic journals have more "memory" about Armstrong's contributions to the theme. Certainly this is not only about the author background, but also in the respect of the expected public, *e.g.* P.C. Fishburn some times did not cite *AP*. Without this simple statistical exercise we could interpret the data in an erroneous way because the number of citations of *AP* is bigger in the set of [45] citations classified as *OA* than that classified in economics group (see table 5).

	Research Area		
	Economics	Psychology	Other Areas
Occurrence of reference to Armstrong and Luce (1956)	19	13	24
Correlation	0.128	0.016	−0.118
t-statistics	2.613	0.325	−2.417
p-value	0.98%	74.53%	1.61%

Table 5: Number and stats about the simultaneous occurrences of citations of *AP* and Luce (1956)
Source: Our elaboration, data from *WoS* and *GS*

6 Concluding Remarks

Although Armstrong became a not so much remembered author in choice theory, his insistence in the theme of the intransitivity of indifference have occasionally been cited by important authors in economics and others areas, and frequently in highly prestigious journals. In addition to this, we observed that his intuitive insights in this theme was confirmed and enlarged in strongly mathematical representations that are the basis of a growth research branch in the last ten years. In this paper we reconstructed this tradition with special emphasis on the influence of Luce's 1956 paper and just supply some evidences of the importance of P. C. Fishburn. We expect that in future researches we can explore in a deeper perspective the work of this highly productive and eclectic author. We also showed that Luce was closer to an economic perspective in his 1956 paper, although he is mainly related with mathematical psychology. Last, but no least we also made here a most complete tribute to the work of Armstrong on choice theory, one that was absent in the history of economic thought.

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