Fixed-income portfolio optimization based on dynamic Nelson-Siegel models with macroeconomic factors for the Brazilian yield curve

Richard Schnorrenberger* Guilherme Valle Moura†

Abstract

The study investigates the statistical and economic value of forecasted yields generated by dynamic yield curve models which incorporate a large macroeconomic dataset. The analysis starts off by modeling and forecasting the term structure of the Brazilian nominal interest rates using several specifications for the dynamic Nelson-Siegel (DNS) framework, suggested by Diebold and Li (2006). The comparison of forecast performance for forecast horizons above three months supports the evidence for the incorporation of one macroeconomic factor that summarizes broad information regarding mainly inflation expectations. In order to assess the economic value of those forecasted yields, a fixed-income portfolio optimization using the mean-variance approach of Markowitz (1952) is performed. The analysis indicate that good yield curve predictions are important to achieve economic gains from forecasted yields in terms of portfolio performance. Preferred forecasted yields for short forecast horizons perform quite well for optimal mean-variance portfolios with one-step-ahead estimates for fixed-income returns, while forecasted yields generated by a macroeconomic DNS specification outperforms in terms of portfolio performance with six-step-ahead estimates. Therefore, there is an economic and statistical gain from considering a large macroeconomic dataset to forecast the Brazilian yield curve dynamics, specially for longer forecast horizons.

Keywords: Fixed-income portfolio optimization. Brazilian yield curve. Dynamic Nelson-Siegel model. Macroeconomic factors. Yield curve forecasting. Mean-variance approach.

Resumo

O estudo investiga o valor estatístico e econômico dos rendimentos previstos por modelos dinâmicos da curva de juros que incorporam um grande conjunto de dados macroeconômicos. A análise parte da modelagem e previsão da estrutura a termo das taxas de juros nominais brasileiras, usando diversas especificações para o modelo dinâmico de Nelson-Siegel (DNS), sugerido por Diebold and Li (2006). A análise comparativa de performance preditiva dos modelos para horizontes de previsão acima de três meses apoia a evidência para a incorporação de um fator macro que resume principalmente informações gerais sobre expectativas de inflação. Para avaliar o valor econômico dos rendimentos previstos, é realizada uma otimização de carteira de renda fixa usando a abordagem de média-variância de Markowitz (1952). A análise indica que boas previsões para as curvas de juros são importantes para obter ganhos econômicos com os rendimentos previstos em termos de desempenho do portfólio. Rendimentos previstos com maior precisão para horizontes de previsão curtos atingem bons resultados para portfólios ótimos que utilizam estimativas de um passo a frente para os retornos de renda fixa, enquanto que rendimentos previstos gerados por uma especificação macroeconômica do modelo DNS atingem bom desempenho para a otimização que utiliza estimativas de doze passos a frente. Portanto, há um ganho econômico e estatístico ao considerar um grande conjunto de dados macroeconômicos para prever a dinâmica da curva de juros brasileira, especialmente para horizontes maiores de previsão.

Palavras-chave: Otimização de portfólio de renda fixa. Curva de juros brasileira. Modelo dinâmico de Nelson-Siegel. Fatores macroeconômicos. Previsão da curva de juros. Abordagem de média-variância.

Códigos JEL: C13; C5; E43; E44; E47.
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*Master Degree in Economics, Universidade Federal de Santa Catarina. Email: richard.schnr@gmail.com
†Department of Economics, Universidade Federal de Santa Catarina. Email: guilherme.moura@ufsc.br

1 Introduction

There is a wide heterogeneity between term structure models that try to fit and forecast the dynamic behavior of yield curves. Traditional term structure models decompose interest rates into a set of yield latent factors, such as level, slope and curvature (Litterman and Scheinkman, 1991). Even providing good in-sample fit (Nelson and Siegel, 1987; Dai and Singleton, 2000) and satisfactory results for out-of-sample forecasts (Duffee, 2002; Diebold and Li, 2006), the economic meaning of such models is limited since they neglect a macroeconomic environment that could affect interest rates of different maturities. Many yield curve models simply ignore macroeconomic linkages. Nonetheless, there are macroeconomic forces that shape the term structure, so that changes in macroeconomic variables can have some impact on future movements of the yield curve (Gürkaynak and Wright, 2012). According to Ang and Piazzesi (2003), the incorporation of macroeconomic information can generate term structure models that forecast better than those without macroeconomy effects, which is of great interest for fixed-income portfolio analytics. Thereby, researchers have begun to use a joint macro-finance modeling strategy, which provides the most comprehensive understanding of the term structure of interest rates.

The development of term structure models that integrate macroeconomic and financial factors is recent in economic research. Ang and Piazzesi (2003), Diebold et al. (2006) and Hördahl et al. (2006) provide the pioneering studies that incorporate macroeconomic information to explain the dynamics of the yield curve through time, and thus representing an active progress to solve the missing linkage between macroeconomy and term structure models. Diebold et al. (2006) provide a macroeconomic interpretation of the DNS model, suggested by Diebold and Li (2006). They combine observable macroeconomic variables, basically related to real activity, inflation, and monetary policy, and yield factors into the Vector Autoregression (VAR) that governs the dynamics of factors. While these studies consistently find significant relationships between macroeconomic variables and government bond yields, they ignore potential macroeconomic information that could be useful for yield curve modeling and forecasting.

More recently, a literature that uses large macroeconomic datasets has emerged, based on the idea that monetary authorities use rich information sets to take monetary policy decisions (Bernanke and Boivin, 2003). Moench (2008) proposed to use the "Factor-Augmented VAR" (FAVAR) (Bernanke et al., 2005) procedure to jointly model the yield curve dynamics and macro factors extracted from a large macroeconomic dataset, taking advantage of systematic information contained within large datasets. De Pooter et al. (2010) and Favero et al. (2012) also use "data-rich environments" for the term structure by extracting common macro factors through dimensionality reduction techniques, such as principal component analysis. In addition, Vieira et al. (2017) combine the FAVAR methodology with the DNS model for the Brazilian yield curve, adding forward-looking variables about the macro-financial scenario into the macroeconomic dataset. In general, these studies consistently reveal that the inclusion of few macro principal components leads to better out-of-sample yield forecasts compared to benchmark models that use individual macro variables or do not incorporate macroeconomic information.

Term structure models play an important role in fixed-income asset pricing, strategic asset allocation and portfolio analytics. The evolution of the yield curve is essential to compute the risk and return characteristics of one's fixed-income portfolio (Bolder, 2015). In order to take an active position in a fixed-income portfolio, based on the mean-variance approach of Markowitz (1952), dynamic yield curve models are used to generate yield forecasts for selected maturities, which are then used to compute expected fixed-income returns. The fixed-income portfolio problem essentially consists in predicting the distribution of returns for a set of maturities and select the optimal vector of portfolio weights conditional on one's expected returns and risk preferences.

Although the mean-variance approach of Markowitz has been widely explored in the context of equity portfolios, little is known about portfolio optimization in fixed-income markets. A recent literature, kick-started by Korn and Koziol (2006), that exploits the risk-return trade-off in bond returns has emerged. Caldeira et al. (2016) perform a mean-variance bond portfolio selection by employing dynamic factor models for the term structure and derive simple closed-form expressions for expected bond returns and their covariance matrix based on forecasted yields. Along with those studies, the present study contributes to validate the use of term structure models to perform mean-variance optimization in the fixed-income context using datasets of Brazilian nominal interest rates.

This study contributes to the present literature by assessing the economic value of forecasted yields generated by yield curve models which incorporate a large macroeconomic dataset. That is, it combines the benefits from incorporating macroeconomic information into term structure models and the use of those forecasted yields to assess their economic value through a portfolio optimization analysis. The incorporation of macro factors into term structure models has the theoretical premise of increasing the model's predictive power. In this sense, the main question is the following: Is there some economic gain, in terms of portfolio performance, from incorporating macroeconomic information into term structure models? Hence, the major purpose is to investigate the magnitude of the statistical and economic gain with the incorporation of a large macroeconomic dataset into the dynamic Nelson-Siegel model.

The empirical evidence indicates that the incorporation of one macro factor, which summarizes broad macroeconomic information regarding mainly inflation expectations, contributes to improve yield curve predictions for 6- and 9-month-ahead forecast horizons, specially for medium and long-term maturities. The conclusion that macroeconomic information tends to improvement in yield curve forecasting extend results found in previous literature. In the context of portfolio selection, good yield curve predictions proved to be important to achieve better results in terms of portfolio performance. Parsimonious yield curve models without macroeconomic information and with better forecast accuracy for short forecast horizons perform quite well for optimal mean-variance portfolios with one-step-ahead estimates for fixed-income returns. On the other hand, forecasted yields generated by a macroeconomic specification provide better information to perform a mean-variance portfolio optimization which uses six-step-ahead estimates for fixed-income returns.

The outline of the study is as follows. Section 2 discusses the theoretical dynamic yield curve models used for the prediction analysis, while Section 3 describes the empirical data and the estimation procedure. Section 4 discusses the empirical results and discussion regarding the out-of-sample forecast exercise for the Brazilian yield curve and reports results for the application of those forecasted yields to fixed-income portfolio optimization. Finally, Section 5 is composed by concluding remarks.

2 The dynamic Nelson-Siegel model

In general, many forces are at work at moving interest rates. Identifying these forces and understanding their impact on yields, is therefore of crucial importance for asset pricing, portfolio analytics and risk management (De Pooter et al., 2010). Term structure models aim to specify the behavior of interest rates, seeking to identify the driving factors that help to explain prices of fixed-income securities.

Diebold and Li (2006) suggested the dynamic Nelson-Siegel model by introducing dynamic components through time-varying parameters to the static Nelson and Siegel (1987) framework. Furthermore, up to Diebold and Li (2006) few term structure models gave importance to out-of-sample forecasting perspective[1]. The mechanics of DNS follow the functional form of Nelson and Siegel (1987), which specifies a linear combination of three smooth exponential factors to adjust the variety of yield curve shapes for a given point in time. The DNS model has a good cross section fit to the observed interest rates at different maturities and incorporates a time-series environment through time-varying factors:

$$y_t(\tau) = \beta_{1t} + \beta_{2t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + \beta_{3t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right). \tag{1}$$

[1] Diebold and Li (2006) argue that equilibrium and arbitrage-free models focus only on fitting the term structure at a given point of time to ensure the absence of arbitrage opportunities. As they seek to incorporate dynamic and the out-of-sample forecast perspective to yield curve, the authors use a model capable to describe the future dynamics of the yields for different maturities over time.

where $y(\tau)$ represents the continuously compounded yield to maturity of a zero coupon bond with maturity τ and maturity value equal to unity, and parameter λ controls the exponential decay rate of the curve, or the rate at which factor loadings decay to zero. DNS carries cross-sectional and time-series perspectives, representing a spatial and temporal linear projection of $y_t(\tau)$ on the time-varying variables β_{1t}, β_{2t} and β_{3t}, which can be interpreted respectively as level, slope and curvature factors of the term structure (Litterman and Scheinkman, 1991). Theoretical analysis regarding the interpretation of yield latent factors (β_{1t}, β_{2t}, β_{3t}) can be found in Diebold and Li (2006).

The class of Nelson-Siegel models has long been a popular choice among central bankers and financial market practitioners, supported by its appealing statistical features concerning smoothness, flexibility and parsimony. According to Diebold and Rudebusch (2013), dynamic factor models provide appealing features because yield data actually display factor structure. Some key reasons prove its statistical appealing: (i) factor structure generally provides a highly accurate empirical description of yield curve data, because just a few constructed factors can summarize bond price information; (ii) statistical tractability, by providing a valuable compression of information, effectively collapsing an intractable high-dimensional modeling situation into a tractable low-dimensional situation. Beyond good fit and good forecast performance of DNS, its simplicity confirms the increasing popularity of the DNS framework.

State-space representation of the DNS model

Diebold and Li (2006) show that it is possible to interpret the DNS model in state-space system format, assuming that the dynamic latent factors are state variables and follow a stochastic first-order VAR. The state space system constituted by the measurement and transition equations can be summarized by the matrix notation:

$$(f_t - \mu) = A(f_{t-1} - \mu) + \eta_t, \tag{2}$$

$$y_t = \Lambda f_t + \varepsilon_t, \tag{3}$$

for $t = 1, ..., T$, where y_t is the $N \times 1$ vector of observed yields for N different maturities τ_i at time t, so that $y_t = [y_t(\tau_1), y_t(\tau_2), ..., y_t(\tau_N)]'$, where τ_1 is the shortest selected maturity and τ_N is the longest; f_t is the $m \times 1$ state vector containing the level (L_t), slope (S_t) and curvature factors (C_t); μ is the factor mean; A is the $m \times m$ state transition matrix; η_t is the $m \times m$ state equation factor disturbances; Λ is the $N \times m$ sensitivity matrix of the measurement equation; and ε_t is the $N \times 1$ vector containing the measurement disturbances.

The measurement equation (3) adds a stochastic error term which relates the set of N yields to the unobserved yield latent factors. So, the factor loadings matrix Λ relates the yield curve dynamic to the constructed factors. The transition equation (2) determines the common factor dynamics as a first-order process. The covariance structure of the white noise transition and measurement disturbances specify that vectors η_t and ε_t are orthogonal to each other and to the initial state:

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim WN \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix} \right], \tag{4}$$

The covariance matrix of measurement disturbances H is assumed to be diagonal, so that the disturbances ε_t of different maturities are uncorrelated [2]. Further, the covariance matrix of transition disturbances Q is not diagonal, so that the disturbances η_t can be correlated in time, allowing for correlated shocks between yield latent factors.

[2] According to Diebold et al. (2006), this assumption is common in the literature for simplifying model estimation by reducing the number of parameters.

Specification of the macroeconomic models

Following Diebold et al. (2006), the introduction of relationship between the components of the yield curve and macroeconomic factors consists simply in incorporating macro factors as exogenous explanatory variables into the state vector and corresponding expansion of the matrices that form the state-space model (2)-(4). The approach here simply replaces the individual macroeconomic variables used in Diebold et al. (2006) by a small number of macroeconomic factors obtained from a large set of possible regressors. Therefore, the model structure only contemplates effects of macro factors to yield factors in future time periods via dynamic interaction in the transition equation.

The assumption that in a DNS environment, the yield curve can be simply decomposed by L_t, S_t, and C_t, remains valid. The three yield factors are all that one needs to explain most yield variation (Diebold and Rudebusch, 2013), so that the inclusion of macro factors will be used to explain the dynamics of the yield latent factors. Thus, macroeconomic factors extracted from a large set of macroeconomic variables are linked to yield factors, so that a kind of two-step DNS procedure is employed. First, macro factors (e.g., broad real activity and broad inflation expectations) are extracted, and then all factors are analyzed in a joint VAR.

The expansion of the *DNS-VAR(3)* model to macroeconomic representations of the DNS form is given by the incorporation of one and two macro factors, denoted by X^1 and X^2, to the state vector. The state vector is now $f_t^{'} = (L_t, S_t, C_t, X_t^1)$ for the model denominated *DNS-VAR(4)*^{1}, $f_t^{'} = (L_t, S_t, C_t, X_t^2)$ for the model denominated *DNS-VAR(4)*^{2} and $f_t^{''} = (L_t, S_t, C_t, X_t^1, X_t^2)$ for the model denominated *DNS-VAR(5)*. Sometimes, these macroeconomic specifications will be regarded as *yields-macro* models. Table 1 summarizes these general DNS specifications used in the estimation procedure.

Table 1: General DNS specifications set.

Model Specification	Factors
DNS-VAR(3)	Level (L_t), Slope (S_t), Curvature (C_t)
DNS-VAR(4)^1	Level (L_t), Slope (S_t), Curvature (C_t) and 1st Macro Factor (X_t^1)
DNS-VAR(4)^2	Level (L_t), Slope (S_t), Curvature (C_t) and 2nd Macro Factor (X_t^2)
DNS-VAR(4)	Comprehends the *DNS-VAR(4)*^{1} and *DNS-VAR(4)*^{2} models
DNS-VAR(5)	Level (L_t), Slope (S_t), Curvature (C_t), 1st Macro Factor (X_t^1) and 2nd Macro Factor (X_t^2)
yields-macro	Comprehends the *DNS-VAR(4)* and *DNS-VAR(5)* models

The inclusion of the $K = 1, 2$ macroeconomic factors is motivated by the principal components analysis, which extract a small number of common factors from a panel series composed by 182 macroeconomic variables. The approach is supported by the set of conditioning information that monetary authorities take into account when deciding interest rates levels. The ordering of the state factors in $f_t^{'}$ and $f_t^{''}$ is performed this way because the information of the yield curve is observed at the beginning of each month. The expansion of the DNS model also requires an appropriate increase in the dimensions of the matrices that form the system (2)-(4), leading to a considerable increase in the number of parameters to be estimated.

3 Estimation methodology and data

3.1 Estimation procedure

The DNS state-space structure represented by (2)-(4) implies that Kalman filter is immediately applicable for optimal filtering of the yield latent factors (Diebold and Rudebusch, 2013). The unobserved state vector f_t and parameters of the system can be estimated by the one-step DNS approach introduced by Diebold et al. (2006). The method estimates the conditional distribution of vector f_t given the set of information contained in the vector of observed variables $Y_t = \{y_1, ..., y_t\}$, building the likelihood function to be maximized. For the macroeconomic DNS structure, the one-step DNS is not absolutely one-step once macroeconomic factors are obtained separately from the state-space estimation. Thus, macro factors primarily extracted from principal component analysis (PCA)[3] are simply combined with yield latent factors in the state transition equation.

For a certain time series $Y_T = \{y_1, ..., y_T\}$, Kalman filter algorithm works recursively for $t = 1, ..., T$ with initial values for the set of unknown parameters collected in θ. The vector θ is composed by parameters of matrices A, Q and H, together with the vector of average factor states μ and parameter λ, which are treated as time-invariant. The estimation of θ uses a numerical optimization method that maximizes the log-likelihood function[4], which is constructed via decomposition of the one-step-ahead prediction error ($v_t = y_t - \Lambda f_{t|t-1}$, where $f_{t|t-1}$ is the expectation for the state vector f_t given the set of information Y_{t-1}):

$$l(\theta) = -\frac{NT}{2} log(2\pi) - \frac{1}{2}\sum_{t=1}^{T} log|F_t| - \frac{1}{2}\sum_{t=1}^{T} v_t^{'} log(F_t^{-1}) v_t, \tag{5}$$

where F_t is the covariance matrix of the one-step-ahead observation forecasts. For the *DNS-VAR(3)* model, the Kalman filter procedure starts with initial values for the set of parameters θ_0 that comes from the two-step DNS approach described in Diebold and Li (2006), and the initial state values are simply their average filtered in two-step DNS approach. For *yields-macro* models, A_0 and B_0 are set to zero, D_0 with respect to yield components comes from the two-step DNS approach while the part related to macro factors is set to zero, and f_0 and λ_0 follow the assumptions made for *DNS-VAR(3)* model. The initial value for λ_0 is calibrated at 0.0726, as suggested by Diebold and Li (2006).

Immediately, if information is available by time t, for the h-step-ahead forecast horizon the construction of out-of-sample forecasts consists simply in forecasting the factors for a given forecast horizon and apply them to the yield curve equation:

$$f_{t+h|t} = E_t(f_{t+h}|y_t, ..., y_1) = \mu + \prod_{j=t+1}^{t+h} A_j \left(f_{t|t} - \mu\right) = \mu + A^h(f_{t|t} - \mu), \tag{6}$$

$$y_{t+h|t} = E_t(y_{t+h}|y_t, ..., y_1) = \Lambda E_t(f_{t+h}|y_t, ..., y_1) = \Lambda\mu + \Lambda(A^h(f_{t|t} - \mu)). \tag{7}$$

Again, the application of Kalman filter is convenient to extract the optimal general h-step-ahead prediction of both the yield factors and the observed yields ($f_{t+h|1:t}$, $y_{t+h|1:t}$).

3.2 Data description

The macroeconomic factors are extracted from a macro panel containing 182 monthly time series for the Brazilian economy. The table containing the macroeconomic panel data, and each preadjustment applied to obtain stationary time series, is not reported to save space and can be provided upon request. The individual variables are classified in various economic categories as follows: money growth (about 6% of the total set of variables), consumption and sales (10.5%), credit (5.5%), employmet, wage and income (9.4%), price (22.5%), production and real activity (18.7%), financial and risk (5%), fiscal (5.5%), and external sector (17%). Regarding the timing of the macro series, it is worth to note that the observation of macroeconomic data by agents only happens after a certain time of the reference month, because several variables take some time to be released. Thereby, the analysis assumes that agents have an expectation or trustworthy proxy about the current macroeconomic scenario.

Most part of the macroeconomic dataset originates from Rossi and Carvalho (2009) and Almeida and Faria (2014), while the forward-looking variables are based on some variables used by Vieira et al. (2017).

[3]PCA takes advantage of the redundancy of information contained within big data, identifying the patterns in data and replacing a group of variables that measure the same phenomenon by a few new variables. The method generates a set of new variables, called principal components, which form a linear combination of the original variables. The usefulness of PCA regards to data dimensionality reduction without much loss of information.

[4]The MATLAB estimation code uses “fminunc” function to optimize the procedure of finding the unknown parameters.

The forward-looking variables refers to market expectations about several key economic variables, available in the weekly market reports published by the Central Bank of Brazil, so-called Focus report. The forward-looking variables considered focus on the mean of market expectations for 1-1.5 year ahead, 2-2.5 year ahead and 3-5 year ahead. This gives a solid information about the future state of the Brazilian macroeconomy.

Time series regarding Brazilian interest rates are removed from the macro dataset to avoid complications that could emerge for the estimation process from not using an arbitrage-free model. Moench (2008) and Koopman and Van der Wel (2013) also remove all variables relating to interest rates, arguing that central banks do not take into account the information contained in yields when making monetary policy decisions.

The yields data consists in monthly observed yields of Brazilian Inter Bank Deposit Future Contract (DI-futuro) negotiated at the Brazilian Mercantile and Futures Exchange (BM&F)[5]. Information about DI-futuro contracts are taken from the first business day of the month in which the contract is due. The interpolated yield curves are obtained by cubic splines interpolation[6], which allows one to convert observed yields into relevant maturities. The present study converts data into the following $N = 14$ different maturities; $\tau =$ 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 36, 42, 48 and 60 months. The estimation procedure is carried out for the in-sample period from 2003:04 to 2012:11, while the predictive analysis is performed for the out-of-sample period from 2012:12 to 2016:03; a total of $T = 116$ in-sample and $S = 40$ monthly out-of-sample observations.

4 Empirical results and discussion

4.1 Preliminary evidence

The analysis uses two macro factors extracted from the large macroeconomic dataset, where both explain about 24.8% of the overall variation in observed variables. The first macro factor (X^1) exceeds 16.6% of the total variance of the original data and correlates mostly with the following economic categories: production and real activity, price, employment, wage and income. The second factor (X^2) accounts for 8.24% of the variation in original data and correlates mostly with real activity variables, external sector and most prominently with inflation expectations. Besides that, individual forward-looking variables are highly correlated with the common macro factors, specially X^1.

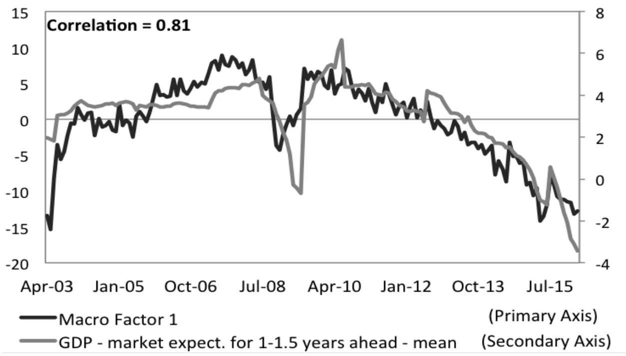
The correlations aforementioned give an indication that X^1 is possibly related to business cycle, while X^2 is likely to represent the price level and central bank's efforts to control inflation. In order to provide some insights about these primary assumptions, Fig. 1 plots the X^1 and X^2 time series against some highly correlated individual macroeconomic variables. The first macro factor exhibited in panels (a) and (b) is a relatively smooth time process that clearly shows characteristics related to business cycle once it strongly correlates with market expectations for GDP and the General Registration of Employed and Unemployed (CAGED). The graph from panel (a) undoubtedly evidence that X^1 follows the path that market expectations are delineating for the one-year-ahead Brazilian economic scenario. For example, the negative values for X^1 in the beginning of 2003, 2008 (recent financial crisis) and after 2013, are clearly associated with recession periods. For these reasons, X^1 can be labelled as *business cycle* factor.

On the other hand, panels (c) and (d) reveal a noisier process for X^2. Factor X^2 presents a relative strong and positive correlation with market expectations for next 12 months IGP-DI and market expectations for 2-2.5 years ahead IPA-M, both inflation indexes. Thus, it is plausible to assume that the second macro factor reflect mainly inflation future scenarios, and can be labelled as *inflation* factor. These common factors extracted here are in line with findings in Koopman and Van der Wel (2013) and De Pooter et al. (2010), where the first macro factor resembles the real activity and the second factor is mostly related to price

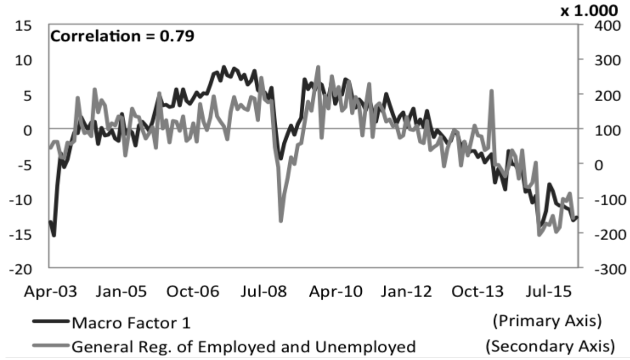
[5]More information on the operation of DI-futuro contracts can be found in Caldeira et al. (2013).

[6]The cubic splines interpolation, proposed by McCulloch (1971), employes piecewise combinations of cubic functions to fit the yield curve. In other words, a piecewise polynomial smoothly connects the yield curve between each pair of vertices (or knot points) of the observed yield data.

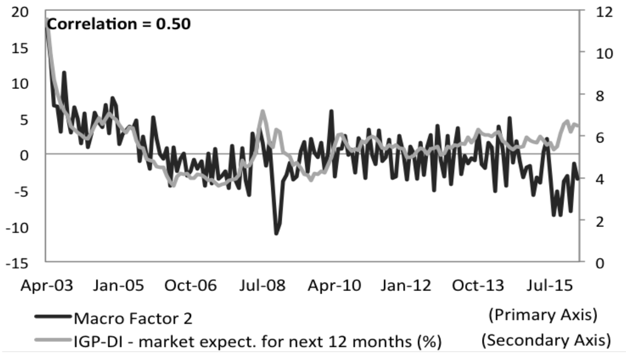
indexes.



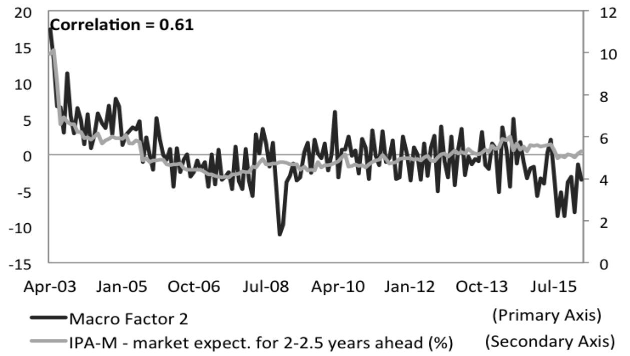
(a)



(b)



(c)



(d)

Figure 1: Plots of common macro factors and individual most correlated macroeconomic variables.

Next, table 2 presents summary statistics for the yield dataset at some selected maturities, including the yield latent factors and the first two standardized macroeconomic factors, dividing between in-sample and out-of-sample periods. As analyzed in Diebold and Rudebusch (2013), several important yield curve facts emerge: (i) time-averaged yields increase with maturity revealing an increasing and slightly concave shape; which reports some kind of term premium, perhaps due to risk aversion, liquidity preferences, or preferred habitats; (ii) yield volatilities decrease with maturity; (iii) yields are highly persistent, as evidenced by the very large 1-month spread autocorrelations, specially for shorter maturities, and by the significant 12-month autocorrelations for the in-sample period. Despite being a period with increasingly shifts in yield curve levels, the out-of-sample period reports lower yield levels in comparison with the in-sample period, and thus, reduced volatility levels.

The in-sample observed yields data show an asymmetric distribution, where most of the observations concentrate in lower rates, while the out-of-sample yields data only show the same behavior for shorter maturities. In addition, the in-sample level factor is skewed to the right due to high observed yields of the first years of the sample period. The slope factor is negative in most part of both sample periods and concentrate in the second quantile of its sample distribution. The observations regarding X^1 pursue areas with positive values, albeit having an absolute minimum value higher than its maximum. So, as X^1 is highly correlated with business cycle, its sample statistics reflect that there are more procyclical periods in the Brazilian economy overall the sample period. The observations of X^2 point to a sample distribution slightly skewed to the left, where data variation is larger for negative values. When it comes to sample autocorrelations, the latent yield factors and X^1 exhibit high persistences at displacement of 1 month, while X^2 shows a moderate persistence for the in-sample period.

Table 2: Summary statistics of yields and macro factors.

	Mean	Sd	Quantiles					$\hat{\rho}(1)$	$\hat{\rho}(12)$
			Min	Q(25%)	Median	Q(75%)	Max		
In-Sample Period Dataset: 2003:04 - 2012:11									
DI-futuro yields (by maturity)									
3	0.1356	0.0410	0.0741	0.1085	0.1233	0.1620	0.2617	0.9415	0.4257
9	0.1354	0.0381	0.0722	0.1097	0.1253	0.1603	0.2441	0.9388	0.4444
15	0.1369	0.0360	0.0745	0.1117	0.1268	0.1623	0.2429	0.9297	0.4333
27	0.1395	0.0337	0.0803	0.1164	0.1297	0.1659	0.2454	0.9136	0.4442
36	0.1406	0.0335	0.0827	0.1177	0.1293	0.1650	0.2516	0.9042	0.4656
60	0.1421	0.0345	0.0866	0.1199	0.1297	0.1639	0.2579	0.9041	0.4893
Yield curve latent factors (level, slope and curvature)									
L_t	0.1459	0.0390	0.0948	0.1205	0.1328	0.1561	0.2913	0.8878	0.4746
S_t	-0.0139	0.0297	-0.0806	-0.0332	-0.0173	0.0005	0.0567	0.8831	-0.0377
C_t	-0.0029	0.0435	-0.1159	-0.0321	-0.0022	0.0287	0.1136	0.8951	0.0058
First two standardized principal components (PC) from the macro series									
1st PC	0	1	-4.3332	-0.5690	0.1103	0.6944	1.6153	0.8249	0.1824
2st PC	0	1	-2.8687	-0.7395	-0.0199	0.5401	4.1118	0.4741	0.3055
Out-of-Sample Period Dataset: 2012:12 - 2016:03									
DI-futuro yields (by maturity)									
3	0.1140	0.0244	0.0703	0.1004	0.1113	0.1401	0.1474	0.9389	0.1129
9	0.1176	0.0240	0.0719	0.1031	0.1149	0.1386	0.1565	0.9260	0.1093
15	0.1197	0.0233	0.0734	0.1077	0.1192	0.1364	0.1613	0.9147	0.0881
27	0.1224	0.0214	0.0800	0.1131	0.1221	0.1329	0.1664	0.8985	0.0364
36	0.1231	0.0211	0.0819	0.1141	0.1230	0.1307	0.1671	0.8943	0.0024
60	0.1243	0.0196	0.0864	0.1173	0.1236	0.1292	0.1662	0.8858	-0.0687
Yield curve latent factors (level, slope and curvature)									
L_t	0.1266	0.0198	0.0925	0.1173	0.1229	0.1346	0.1689	0.8658	-0.1762
S_t	-0.0144	0.0172	-0.0395	-0.0281	-0.0177	-0.0008	0.0269	0.8036	-0.0916
C_t	-0.0041	0.0282	-0.0606	-0.0243	-0.0073	0.0243	0.0539	0.6962	-0.0750
First two standardized principal components (PC) from the macro series									
1st PC	0	1	-1.8151	-0.9609	0.2068	0.8576	1.6432	0.8406	0.1282
2st PC	0	1	-2.0911	-0.5877	0.0283	0.7151	2.0193	0.1160	-0.0394

Notes: The table presents the descriptive statistics for DI-futuro contracts over the in-sample and out-of-sample periods. The monthly yield curves were constructed using cubic splines interpolation. For each maturity, the table displays the mean, standard deviation (Sd), minimum (Min), 25% quantile, median, 75% quantile, maximum (Max), and sample autocorrelations at displacements of 1 ($\hat{\rho}(1)$) and 12 ($\hat{\rho}(12)$) months. In addition, it shows the statistics for the yield latent factors and for the first two standardized principal components extracted from the macro dataset.

4.2 Estimating term structure models

In-sample estimates

The results obtained from estimating the DNS specification models represented by *DNS-VAR(3)* and the set of *yields-macro* models, defined in Section 2, indicate that on average the estimated models provide a good fit to the yield curve across the entire maturity spectrum, except for very short maturities. For maturities above 9 months the models fit the observed data efficiently well. The bad fit behavior for short maturities also is reported by Diebold et al. (2006), where estimated errors for yields of 3-months maturity are relatively higher. In addition, yield curve estimates of the macroeconomic specifications for medium and long-term maturities are more accurate than *DNS-VAR(3)* estimates. Furthermore, the estimated errors of *DNS-VAR(5)* model are higher compared to *DNS-VAR(4)*[2] model. These results sign for a path where macroeconomic information can improve yield curve predictions, at least for longer maturities.

Out-of-sample estimates

In this section, we perform the out-of-sample forecast exercise using a rolling window analysis. This implies that the multiple step ahead forecasts explored here are closely related to an investor's pseudo real-time decision. However, the analysis is not based on fully real-time data once some macroeconomic variables are constructed from the revised dataset and some macro information have not been released yet at the time when a forecast is made.

The forecast exercise for the multiple forecast horizons of 1-, 3-, 6-, 9- and 12-month-ahead are performed with rolling window samples of size $T = 116$. Hence, the out-of-sample forecasts are carried out over the time interval from December 2012, to March 2016. Predictions are made for $T + h$ at the end of each rolling window, where h is the forecast horizon. The number of rolling window samples is $S = 40$, whereas the last 11 rolling window samples have some restrictions related to forecast horizons. That is, there are 40 out-of-sample forecasts for 1-month horizon, 39 for 2-month horizon, and so on until 29 out-of-sample forecasts for 12-month horizon.

The evaluation of out-of-sample forecasts requires some measures to compute the errors for each maturity τ_i. Given a time series of S out-of-sample forecasts for h-period-ahead forecast horizon, the root mean squared forecast error (RMSFE) calculates a forecast error measure for maturity τ_i at forecast horizon h and for model m:

$$\text{RMSFE}_m(\tau_i) = \sqrt{\frac{1}{S}\sum_{t=1}^{S}(\hat{y}_{t+h|t}(\tau_i) - y_{t+h}(\tau_i))^2}, \tag{8}$$

where $y_{t+h}(\tau_i)$ is the yield for the maturity τ_i observed at time $t + h$, and $\hat{y}_{t+h|t}(\tau_i)$ is the corresponding forecast made at time t. The performance analysis also reports the trace root mean squared forecast error (TRMSFE), which calculates the trace of the covariance matrix of the forecast errors across all N maturities:

$$\text{TRMSFE}_m(\tau_i) = \sqrt{\frac{1}{N}\frac{1}{S}\sum_{i=1}^{N}\sum_{t=1}^{S}(\hat{y}_{t+h|t}(\tau_i) - y_{t+h}(\tau_i))^2} \tag{9}$$

The Giacomini and White (2006) test is applied to compare forecast accuracy between two competing models. The Giacomini-White (GW) statistic is a test of conditional forecasting ability, handling forecasts based on both nested and non-nested models, and is constructed under the assumption that forecasts are generated using a moving data window. In this case, the GW test evaluates whether the out-of-sample forecast error from the random walk (RW) model for maturity τ_i $(e^{RW}_{t+h|t}(\tau_i))$ is statistically different from the forecasts of the competing DNS model. The test is based on the loss differential function $d_{m,t} = (e^{RW}_{t+h|t}(\tau_i))^2 - (e^{DNS}_{t+h|t}(\tau_i))^2$. Thus, we assume a quadratic loss function, where the null hypothesis of equal forecasting accuracy can be written as

$$H_0 : E[d_{m,t+h}|\delta_{m,t}] = 0. \tag{10}$$

Parameter $\delta_{m,t}$ is a $p \times 1$ vector of test functions. The GW test statistic $GW_{m,t}$ can be computed as the Wald statistic:

$$GW_{m,t} = S\left(S^{-1}\sum_{t=T+1}^{S-h}\delta_{m,t}d_{m,t+h}\right)'\hat{\Omega}_S^{-1}\left(S^{-1}\sum_{t=T+1}^{S-h}\delta_{m,t}d_{m,t+h}\right) \xrightarrow{d} \chi^2_{dim(\delta)}, \tag{11}$$

where $\hat{\Omega}_S^{-1}$ is a consistent HAC estimator for the asymptotic variance of $\delta_{m,t}d_{m,t+h}$, and S the number of out-of-sample observations. Under the null hypothesis given in 10, the test statistic $GW_{i,t}$ is asymptotically distributed as χ^2_p.

Table 3 reports the summary statistics of forecast performance for the general specifications: *DNS-VAR(3)*, *DNS-VAR(4)* and *DNS-VAR(5)*. The following basic considerations can be made: (i) the *DNS-*

Table 3: (Trace)-Root Mean Squared Forecast Errors of *DNS-VAR(3)* and *yields-macro* models.

	Panel A: *DNS-VAR(3)* model					**Panel B: *DNS-VAR(5)* model**				
Maturities	Forecast horizon					Forecast horizon				
	1-M	3-M	6-M	9-M	12-M	1-M	3-M	6-M	9-M	12-M
3	0.486*	0.969*	1.817	2.472	2.978	1.424	2.422	3.357	6.536	14.204
6	0.478*	0.987	1.761	2.313	2.739	1.491	2.435	3.275	6.217	13.395*
9	0.502*	1.021	1.691	2.185	*2.581**	1.565	2.487	3.211	5.936	12.710*
12	0.536	1.043	1.641	*2.112*	*2.468**	1.622	2.508	3.166	5.705	12.141*
15	0.546*	1.049	1.610	*2.058**	*2.381**	1.654	2.530	3.112	5.482	11.585*
18	*0.574**	*1.084**	*1.620*	*2.033**	*2.329**	1.687	2.542	3.048	5.239	11.063*
21	*0.600**	*1.114*	*1.628*	*2.020**	*2.287**	1.708	2.537	2.985	5.009	10.565*
24	*0.615**	*1.125*	*1.643*	*2.020**	*2.264**	1.717	2.527	2.942	4.805	10.090*
27	*0.617**	*1.132*	*1.654*	*2.005**	*2.250**	1.718	2.519	2.912	4.608	9.640*
30	*0.622**	*1.139*	*1.652*	*1.994**	*2.241**	1.722	2.511	2.891	4.422	9.212*
36	*0.624**	*1.134*	*1.658*	*2.001**	*2.242**	1.728	2.511	2.838	4.112	8.452*
42	*0.630**	*1.138*	*1.675*	*2.012**	*2.257**	1.726	2.507	2.818	3.880	7.811*
48	*0.633**	*1.132*	*1.670*	*2.012**	*2.262**	1.717	2.510	2.816	3.699	7.292*
60	*0.640**	*1.125*	*1.670*	*2.023**	*2.267**	1.727	2.550	2.869	3.519	6.624*
TRMSFE	0.581	*1.087*	*1.672*	*2.094*	*2.406*	1.660	2.507	3.022	5.026	10.581

	Panel C: *DNS-VAR(4)*^{1} model					**Panel D: *DNS-VAR(4)*^{2} model**				
Maturities	Forecast horizon					Forecast horizon				
	1-M	3-M	6-M	9-M	12-M	1-M	3-M	6-M	9-M	12-M
3	1.864	2.925	4.304	5.949*	8.216*	0.523*	1.069	**1.724***	**2.202**	***2.483****
6	1.834*	2.936	4.251*	5.823*	7.924*	0.520*	1.044	**1.638**	***2.015***	***2.285***
9	1.793*	2.920*	4.185*	5.706*	7.688*	0.556	1.043	**1.560**	***1.894***	***2.121***
12	1.773*	2.908*	4.103*	5.595*	7.430*	0.592	1.051	***1.520***	***1.798***	***2.010****
15	1.738*	2.873*	4.020*	5.454*	7.215*	0.616*	**1.049***	***1.484***	***1.731****	***1.912****
18	1.714*	2.848*	3.929*	5.323*	7.001*	0.645*	***1.067****	***1.484***	***1.695****	***1.841****
21	1.687*	2.803*	3.853*	5.206*	6.793*	0.665*	***1.090****	***1.489***	***1.669****	***1.787****
24	1.661*	2.763*	3.784*	5.087*	6.614*	0.677*	***1.095****	***1.495****	***1.647****	***1.760****
27	1.638*	2.725*	3.697*	4.979*	6.459*	0.680*	***1.102***	***1.497****	***1.641****	***1.757****
30	1.612*	2.694*	3.607*	4.884*	6.322*	0.687*	***1.106****	***1.502****	***1.636****	***1.756****
36	1.594	2.617*	3.511*	4.712*	6.077*	0.703*	***1.121***	***1.510****	***1.639****	***1.765****
42	1.571	2.564*	3.429*	4.572*	5.889*	0.709*	***1.125***	***1.520****	***1.657****	***1.786****
48	1.548	2.510*	3.347*	4.455*	5.724*	0.711*	***1.122***	***1.515****	***1.661****	***1.797****
60	1.518	2.431*	3.228	4.275*	5.468*	0.715*	***1.121***	***1.516****	***1.677****	***1.813****
TRMSFE	1.685	2.756	3.818	5.169	6.822	0.646	***1.086***	***1.534***	***1.762***	***1.932***

Notes: The table presents the forecasting performances of *DNS-VAR(3)* model and *yields-macro* models. In particular, it reports the RMSFE and TRMSFE statistics obtained by using individual *DNS-VAR(3)*, *DNS-VAR(4)*^{1}, *DNS-VAR(4)*^{2} and *DNS-VAR(5)* models. The values reported are divided by 1×10^{-2}. The RMSFE is reported for each model for the τ maturities and for 1-, 3-, 6-, 9- and 12-month-ahead forecast horizons. The latest line of each panel reports the TRMSFE for the different forecast horizons. Numbers in **bold** indicate that the alternative *yields-macro* models from panels B, C and D outperform the *DNS-VAR(3)* model, otherwise indicate underperformance. Number in *italic* indicate that the DNS model outperform the random walk model. The star on the right of the cell entries indicate where the GW test rejects the null of equal forecasting accuracy between the competitor *yields-macro* models and random walk, with 10% probability of the null hypothesis.

VAR(5) and *DNS-VAR(4)*^{1} clearly underperform the general DNS framework for the entire maturity and forecast horizon spectrum; (ii) the *DNS-VAR(4)*^{2} consistently outperform the *DNS-VAR(3)* model for most maturities and for forecast horizons longer than one month. The GW test rejects the null hypothesis at a 10% level of the *DNS-VAR(4)*^{2} model for some particular cases: (i) 3-month-ahead predictions at medium-term maturities; (ii) 6-month-ahead predictions for maturities above 24 months; and (ii) 9- and 12-month-ahead predictions for the medium and long end of the yield curve. Therefore, the *DNS-VAR(4)*^{2} model generates better forecasts for forecast horizons longer than one month at medium- and long-term maturities.

The GW test also rejects the null for most forecasted yields of *DNS-VAR(4)*^{1} model and few 12-month-ahead forecasted yields of the *DNS-VAR(5)* model, confirming their inferior performance in relation to the RW and *DNS-VAR(3)* models. Both specifications which include the business cycle factor forecast poorly, supporting the evidence of relatively small impact of X^1 on the Brazilian yield curve. In other words, the incorporation of macro factors containing information strongly correlated with business cycle do not contribute to predict the Brazilian yield curve.

The forecasts produced by the *DNS-VAR(4)*^{2} model provide the lowest RMSFEs and TRMSFEs for most predictions above 1-month horizon, while *DNS-VAR(3)* and RW provide the lowest values for 1-month-ahead forecasts. Thus, the inclusion of the inflation factor into the general DNS framework appears to lead to lower RMSFEs for most interest rates and most forecast horizons above 1 month, where results for the GW tests indicate significant improvements from forecasting with *DNS-VAR(4)*^{2} in relation to RW. Overall, the results imply the support for the incorporation of a macro factor that summarizes broad macroeconomic information regarding mainly inflation expectations. The forecast exercise confirms the estimates reported by Moench (2008), De Pooter et al. (2010), Koopman and Van der Wel (2013), Almeida and Faria (2014), among other studies which report better out-of-sample yield forecasts for term structure models with macroeconomic appeal.

4.3 Application to fixed-income portfolio optimization

The mean-variance portfolio problem

The approach suggested by Markowitz (1952) is the most common formulation of portfolio choice problems, which point out that investors allocate their wealth in risky assets based on the trade-off between expected return and risk. At the moment of the portfolio choice, it is assumed that investors are only concerned with the expected returns for the h-step-ahead forecast horizon and its covariance matrix, defined by $\mu_{r_{t|t-h}}$ and $\Sigma_{r_{t|t-h}}$, respectively. The mean-variance portfolio problem can then be formulated by minimizing the portfolio variance for a particular h-step-ahead expected return, subject to additional restrictions on the vector of optimal weights w_t:

$$\underset{w_t}{\text{Min}} \quad w_t^{'} \Sigma_{r_{t|t-h}} w_t - \frac{1}{\delta} w_t^{'} \mu_{r_{t|t-h}}$$

$$\text{subject to}: \; w_t^{'} \imath = 1; \; w_t \geq 0. \tag{12}$$

where $\imath$ is an appropriately sized vector of ones and δ is the investor's risk aversion coefficient. Vector $\mu_{r_{t|t-h}}$ collects the h-step-ahead expected returns for maturities $\tau_1, ..., \tau_N$, so that its dimension is $N \times 1$, while the covariance matrix $\Sigma_{r_{t|t-h}}$ is $N \times N$. The optimization problem is subject to both constraints, the non-negative individual weights, which restricts short sales, and the budget constraint, which ensures that all wealth is invested in risky assets. As the mean-variance problem solves a quadratic utility function, the necessary and sufficient condition for optimization is to solve the optimal weights w_t for the first order condition.

The distribution of log-returns

Following the discussion in Caldeira et al. (2016), factor models for the term structure of interest rates are designed only to model bond yields. Thus, the forecasting stage of yield curve models aim for modeling merely moments of the expected yields. However, the fixed-income portfolio problem requires estimates of the expected return of each maturity, as well as estimates of their covariance matrix. The following mathematical decompositions show that it is possible to obtain expressions for the expected return of fixed-income securities and their covariance matrix based on the distribution of the expected yields.

The mean-variance portfolio optimization is performed for two different forecast horizons: (i) first, one-step-ahead forecasts for log-returns of DI-futuro contracts are used to optimize fixed-income portfolios with monthly rebalancing; and subsequently (ii) six-step-ahead forecasts for log-returns are used to find optimal portfolios with biannual rebalancing. For this reason, the portfolio choice problem requires moments of the expected yields for one- and six-months-ahead forecast horizons. The system of Eqs. (2)-(4) implies that the distribution of one-step-ahead forecasts for y_t, of any maturity τ_i, is normally distributed, i.e.

$y_{t|t-1} \sim N(\mu_{y_{t|t-1}}, \Sigma_{y_{t|t-1}})$, with moments given by[7]:

$$\mu_{y_{t|t-1}} = E_{t-1}[y_t] = \Lambda f_{t|t-1}, \tag{13}$$

and

$$\Sigma_{y_{t|t-1}} = \Lambda(AP_{t-1|t-1}A' + Q)\Lambda' + H, \tag{14}$$

where $P_{t|t}$ denotes the filtered covariance matrix of $f_{t|t}$ at time t. Eq. (13) follows straightforward from Eq. (7), which define the expectation of yields for the h-step-ahead forecast horizon. Note that the covariance matrix of the true, but non-observable states (f_t), would be given simply by Q. However, as stated in Eq. (13), predicted states based on filtered estimates of f_{t-1} are used when computing expected yields. Thus, Eq. (14) takes into account the uncertainty in the Kalman filter estimates of the unobserved factors through $AP_{t-1|t-1}A'$, containing the covariance matrix of the filtered states ($P_{t-1|t-1}$) and not only the covariance matrix of the unobserved factors, Q[8]. Therefore, the first term in Eq. (14), $AP_{t-1|t-1}A' + Q$, adjusts for the fact that filtered estimates are used in (13), and not the true value of states.

Similarly, the distribution of six-step-ahead forecasts, $y_{t|t-6} \sim N(\mu_{y_{t|t-6}}, \Sigma_{y_{t|t-6}})$, is normally distributed with moments given by:

$$\mu_{y_{t|t-6}} = E_{t-6}[y_t] = \Lambda f_{t|t-6}, \tag{15}$$

and

$$\Sigma_{y_{t|t-6}} = \Lambda(A^6 P_{t-6|t-6}A'^{\,6} + \sum_{i=1}^{6} A^{i-1}QA'^{\,i-1})\Lambda' + H. \tag{16}$$

Note that, in this case, the term $(A^6 P_{t-6|t-6}A'^{\,6} + \sum_{i=1}^{6} A^{i-1}QA'^{\,i-1})$ adjusts for the fact that the model uses filtered estimates for the six-step-ahead forecasts of yields and accumulates the uncertainty in the Kalman filter estimates for each step forecast.

Using the mathematical expression for the discount curve $P_t(\tau_i) = \exp(-\tau_i \cdot y_t(\tau_i))$ and the log-return expression, the realized return, $r_t(\tau_i)$, of holding a security from $t-h$ to t while its maturity decreases from τ_i to τ_{i-h}, can be computed as follows,

$$r_t(\tau_i) = \log\left(\frac{P_t(\tau_{i-h})}{P_{t-h}(\tau_i)}\right) = \log P_t(\tau_{i-h}) - \log P_{t-h}(\tau_i) = -\tau_{i-h} \cdot y_t(\tau_{i-h}) + \tau_i \cdot y_{t-h}(\tau_i). \tag{17}$$

It is clear to note from (13)-(17) that the vector of h-step-ahead forecasts of log-returns, $r_{t|t-h}$, also follows a Normal distribution $N(\mu_{r_{t|t-h}}, \Sigma_{r_{t|t-h}})$ with mean given by:

$$\mu_{r_{t|t-h}} = -\tau_{i-h} \otimes \mu_{y_{t|t-h}}(\tau_{i-h}) + \tau_i \otimes y_{t-h|t-h}(\tau_i), \tag{18}$$

where $\mu_{y_{t|t-h}}(\tau_{i-h})$ is the mean vector of the expected yields with maturity τ_{i-h} at time t conditional on period $t-h$ information, $y_{t-h|t-h}(\tau_i)$ is the vector of observed yields with maturity τ_i at time $t-h$, and $\otimes$ represents the Hadamard (elementwise) multiplication. The conditional covariance matrix of the expected log-returns, which is positive-definite, is given by:

$$\Sigma_{r_{t|t-h}} = \tau'_{i-h}\tau_{i-h} \otimes \Sigma_{y_{t|t-h}}. \tag{19}$$

The discussion above solves the problem for obtaining estimates of the expected log-returns for fixed-income securities and their covariance matrix based on yield curve models such as the DNS model, which are essential inputs to the portfolio choice problem based on the mean-variance approach suggested

[7](Durbin and Koopman, 2012, p. 112) define the general formulations for the conditional mean square error matrix and conditional mean of the covariance matrix of predicted states.

[8]For comparison, Caldeira et al. (2013) use the true value of the state vector and show that the second moment of $y_{t|t-1}$ just takes into account the covariance matrix of the unobserved factors, Q.

by Markowitz. All ingredients necessary to calculating the closed-form expressions (18)-(19) are easily retrieved from the Kalman filter estimation discussed in Section 3.

Methodology for evaluating portfolio performance and implementation details

This section aims to assess the economic value of the forecasting ability of the major yield curve models estimated previously. The empirical implementation of the mean-variance optimization problem defined by (12) is performed by using one- and six-step-ahead estimates of the vector of expected returns and its covariance matrix considering five alternative values for the risk aversion coefficient δ: 0.0001, 0.01, 0.1, 0.5 and 1. Following the recursive estimation strategy of the yield curve models, the optimal mean-variance portfolios are also computed recursively as new h-step-ahead estimates for DI-futuro returns are known. Moreover, optimal mean-variance portfolios using one-step-ahead forecasts for DI-futuro returns are rebalanced on a monthly basis, while portfolios using six-step-ahead forecasts are rebalanced on a biannual basis. Thus, the empirical analysis with monthly rebalancing computes the optimal portfolio for each period over the S out-of-sample observations giving a sample of 40 optimal portfolio weights w_t.

Otherwise, the optimization with biannual rebalancing computes the optimal portfolio each six consecutive months. Nevertheless, the portfolio performance statistics are computed for every month of the out-of-sample period. The last rebalancing procedure is performed at May 2015, because as from September 2015, there are no forecasts for 6-month-ahead yields being considered by the yield curve models. Moreover, rebalancing frequency is important when dealing with fixed-income assets, because the securities in the portfolio can age and be closer to maturity. For this reason, the shortest maturity considered here is $\tau_i = 9$ months, because DI-futuro contracts with maturity lower or equal to 6 months will already be matured before the subsequent rebalancing process. This only allows the computation of performance statistics until January 2016, because at February 2016 the 9-month security will already be matured, so that the number of out-of-sample observations is equal to 38.

It is clear that the scenario with biannual rebalancing requires more diligence regarding implementation procedure and computation of performance statistics. Note that, after one period, an optimal portfolio containing securities that yield an average duration of τ_i at the time of the mean-variance optimization, becomes a portfolio with average duration τ_{i-1} and so on until the next rebalancing process, changing the characteristics of the original portfolio over time. Thus, the computation of the time series of portfolio returns need to take care about the constant decrease of the time-to-maturity of its securities.

The performance analysis use some alternative criteria to evaluate the performance of the optimal mean-variance fixed-income portfolios. First of all, we describe the evaluation criteria related to portfolio excess return relative to the risk-free rate (R_{f_t}), which is consider to be the short Brazilian Interbank Deposit (CDI) rate. The average gross (i.e., before transaction costs) excess return relative to the risk-free rate ($\bar{rx}$) is calculated as follows:

$$\bar{rx} = \frac{1}{S} \sum_{t=1}^{S} rx_t,$$

where $rx_t = w_{t-1}^{'} R_t - R_{f_t}$ denotes the gross excess portfolio return at time t and $R_t = [r_t(\tau_i), ..., r_t(\tau_N)]'$ is a vector collecting DI-futuro returns of all maturities considered.

According to Han (2006), it would be appropriate to consider transaction costs when rebalancing the portfolio weights frequently. The empirical scenario with biannual rebalancing can alleviate the impact of transaction costs on portfolio performance. However, the less frequent rebalancing means that the portfolio weights will be outdated, which could negatively affect the portfolio performance because investors would be investing away from the optimal one (Caldeira et al., 2016). The performance analysis also considers the excess return net of transaction costs (rx_t^{net}), which takes into account the negative impact of transaction costs on portfolio average performance, and is calculated as:

$$rx_t^{net} = (1 - c \cdot turnover_t)(1 + rx_t) - 1, \tag{20}$$

where c is the fee that must be paid for each transaction and $turnover_t$ is the portfolio turnover at time t, defined as the fraction of wealth traded between periods $t-1$ and t, i.e,

$$turnover_t = \sum_{i=1}^{N}(|w_{i,t} - w_{i,t-1}|).$$

The parameter $w_{i,t}$ is the optimal weight of maturity τ_i at time t. The level of transaction costs being considered is 5 bps, reflecting a fixed percentage for each rebalance trade. Similarly to $\bar{rx}$, the average excess portfolio return net of transaction costs is defined as $\bar{rx}^{net} = \frac{1}{S}\sum_{t=1}^{S} rx_t^{net}$. Moreover, statistics regarding the volatility (standard deviation) of the net excess return ($\hat{\sigma}$) and the risk-adjusted net excess return (SR) measured by the Sharpe ratio are calculated as follows,

$$\hat{\sigma} = \sqrt{\frac{1}{S}\sum_{t=1}^{S}(rx_t^{net} - \hat{\mu_p})^2},$$

$$SR = \frac{\bar{rx}^{net}}{\hat{\sigma}},$$

where $\hat{\mu_p}$ denotes the sample mean of the portfolio net excess return. Ultimately, the performance analysis takes into account the average duration in years of the portfolios, which allows one to better understand the composition of the optimal portfolios. The average duration of a fixed-income portfolio is calculated as $\frac{1}{S}\sum_{t=1}^{S} w_t^{'}\tau$, where here τ regards to the vector of individual security durations. A higher (lower) average portfolio duration suggests that the optimal portfolio is invested in long (short) maturities.

Results for mean-variance portfolios

Table 4 reports the out-of-sample performance of mean-variance portfolios of DI-futuro contracts that use estimates of yields from the random walk (RW)[9], *DNS-VAR(3)* and *DNS-VAR(4)*^{2} model specifications. For the scenario which considers one-step-ahead estimates and more frequent portfolio rebalancing (Panel A in Table 4), the overview indicates that positive excess return statistics are essentially obtained when the risk aversion coefficient is higher than 0.01, where the annualized net excess returns range from 0.40% to 1.57%. The best overall performance in terms of Sharpe ratio is achieved by the mean-variance portfolio obtained with the RW model with $\delta = 0.5$ ($SR = 0.472$). When lower risk aversion is considered, most of the results indicate negative Sharpe ratios and higher volatility levels. As expected, an increase in the risk aversion coefficient leads to decreases in portfolio volatility as well as decreases in the average duration, that is, optimal portfolios are invested in short-term maturities. This result is intuitive since lower maturity securities are less risky, allowing investors with higher risk aversion to lower portfolio risk by investing in shorter maturities. This evidence is even more pronounced for the RW model, which quickly decreases volatility and duration with the increase of δ, investing basically in 3- and 6-month maturities for δ's higher than 0.1. For instance, the average portfolio duration across specifications for an investor with risk aversion coefficient $\delta = 1$ is 0.89 year, whereas the same indicator for an investor with $\delta = 1 \times 10^{-4}$ is 2.37 years.

On the other hand, the scenario which considers six-step-ahead estimates for DI-futuro returns and a biannual portfolio rebalancing (Panel B in Table 4) reports negative net excess returns across all model specifications and across all levels of the risk tolerance considered, except for the *DNS-VAR(4)*^{2} model with $\delta < 1$. The best overall performance are achieved by the *DNS-VAR(4)*^{2} model for the smallest δ: $\bar{rx}$ equal to 1.075%, $\bar{rx}^{net}$ =1.065%, volatility (measured by the standard deviation) equal to 10.75%, $SR = 0.099$ and average duration equal to 1.55 years. The *DNS-VAR(4)*^{2} model also minimizes losses for the higher level of δ. The general results show that annualized net excess returns range from -3.153% to 1.065%, and the

[9]It is noteworthy that the covariance matrix of the expected log-returns obtained from forecasted yields of the RW model is simply the sample covariance from the in-sample observations.

Table 4: Performance of optimal DI-futuro contracts mean-variance portfolios.

Yield Curve Model	Mean gross exc. R (%)	Mean net exc. R (%)	Std. Dev. (%)	Sharpe Ratio	Duration (years)
Panel A: one-step-ahead estimates with monthly rebalancing					
δ=0.0001					
Random Walk	-1.143	-1.218	21.981	-0.055	2.600
DNS-VAR(3)	0.880	0.820	23.134	0.035	2.356
DNS-VAR(4)^2	-0.905	-0.983	24.664	-0.040	2.163
δ=0.01					
Random Walk	1.081	1.008	13.040	0.077	1.615
DNS-VAR(3)	0.108	0.042	22.330	0.002	2.261
DNS-VAR(4)^2	-0.897	-0.975	24.642	-0.040	2.162
δ=0.1					
Random Walk	1.135	1.092	4.463	0.245	0.626
DNS-VAR(3)	-0.527	-0.584	21.167	-0.028	1.978
DNS-VAR(4)^2	-0.126	-0.195	21.282	-0.009	1.919
δ=0.5					
Random Walk	0.521	0.511	1.083	0.472	0.287
DNS-VAR(3)	0.574	0.524	16.069	0.033	1.416
DNS-VAR(4)^2	0.531	0.459	18.076	0.025	1.434
δ=1					
Random Walk	0.405	0.401	0.889	0.451	0.261
DNS-VAR(3)	1.622	1.572	13.544	0.116	1.187
DNS-VAR(4)^2	1.052	0.987	15.382	0.064	1.227
Six-step-ahead estimates with semi-annual rebalancing					
Yield Curve Model	Mean gross exc. R (%)	Mean net exc. R (%)	Std. Dev. (%)	Sharpe Ratio	Duration (years)
δ=0.0001					
Random Walk	-3.145	-3.153	23.729	-0.133	3.237
DNS-VAR(3)	-0.093	-0.106	15.551	-0.007	2.263
DNS-VAR(4)^2	1.075	1.065	10.752	0.099	1.553
δ=0.1					
Random Walk	-1.379	-1.388	8.921	-0.156	1.551
DNS-VAR(3)	-0.093	-0.106	15.551	-0.007	2.263
DNS-VAR(4)^2	0.499	0.489	7.546	0.065	1.277
δ=0.5					
Random Walk	-0.127	-0.134	4.432	-0.030	1.008
DNS-VAR(3)	-0.443	-0.457	6.647	-0.069	1.359
DNS-VAR(4)^2	0.159	0.150	5.544	0.027	1.026
δ=1					
Random Walk	-0.141	-0.143	2.984	-0.048	0.806
DNS-VAR(3)	-0.506	-0.518	5.065	-0.102	1.194
DNS-VAR(4)^2	-0.106	-0.111	3.616	-0.031	0.896

Notes: Performance statistics for mean-variance portfolios using the random walk, *DNS-AR(3)*, *DNS-VAR(3)* and *DNS-VAR(4)*^{2} model specifications to compute the forecasted yields for the out-of-sample period. Panel A reports the statistics for the portfolio optimization using one-month-ahead estimates for DI-futuro returns, while Panel B reports the statistics using six-months-ahead estimates. The statistics of gross and net excess returns, standard deviation, and Sharpe ratio are annualized and the average portfolio duration is measured in years. Parameter δ denotes the value of the risk aversion coefficient.

annualized standard deviation ranges from 2.98% to 23.72%, whereas the Sharpe ratio ranges from -0.156 to 0.099. As before, an increase in the risk aversion coefficient leads to decreases in the average duration, indicating optimal portfolios invested mostly in long-term maturities for lower levels of δ. Moreover, the impact of transaction costs is relatively small for estimates with biannual portfolio rebalancing, whereas net excess returns are very close to gross excess returns.

The key difference compared to Panel A concerns the average portfolio duration: it is higher across all model specifications and δ's for the estimates with annual rebalancing; e.g., the average portfolio duration across specifications for $\delta = 1$ is now 0.96 years and for $\delta = 1 \times 10^{-4}$ is 2.35. The comparison also suggests that in general portfolio strategies, optimal mean-variance portfolios with monthly rebalancing deliver higher net excess returns than those with annual rebalancing, pointing out a gain in rebalancing the portfolio weights frequently to keep optimal allocation updated.

Table 4 shows that negative net excess returns prevail in most optimal mean-variance portfolios. In rising interest rate environments, as the out-of-sample period, fixed-income prices suffer from the increase in interest rates in the short term, i.e, rising rate environments can result in negative fixed-income returns. A bond's total return comprises not just price changes, but also income, so that the income on a bond can help offset falling prices, cushioning the overall total return. It turns out that optimal mean-variance portfolios can not benefit from increased yields over the long term because of rebalancing: investors do not hold fixed-income securities until their maturity, which turns portfolio's total returns highly dependent on price changes. For this reason, the income returns are not enough to offset the price decline in DI-futuro contracts.

At least, Fig. 2 illustrates the performance of the optimal DI-futuro mean-variance portfolios by plotting the cumulative net returns over the out-of-sample period obtained with the alternative specifications when δ is equal to 1×10^{-4} and 1. The figure suggests that *DNS-VAR(3)* delivers better performance for mean-variance portfolios using one-step-ahead estimates for returns and for $\delta = 1 \times 10^{-4}$ during most part of the out-of-sample period. For $\delta = 1$, the RW model reports more stable and higher cumulative returns. However, *DNS-VAR(4)*^{2} displays better portfolio performance using six-step-ahead estimates. The best overall performance in terms of cumulative net returns are achieved by mean-variance portfolios obtained with the RW model for $\delta = 1$ and for one-step-ahead estimates (33.17%) and with the *DNS-VAR(4)*^{2} for $\delta = 1 \times 10^{-4}$ and six-step-ahead estimates (33.85%). Hence, the incorporation of macroeconomic information has positive effects to achieve better portfolio performance at the biannual rebalancing scenario.

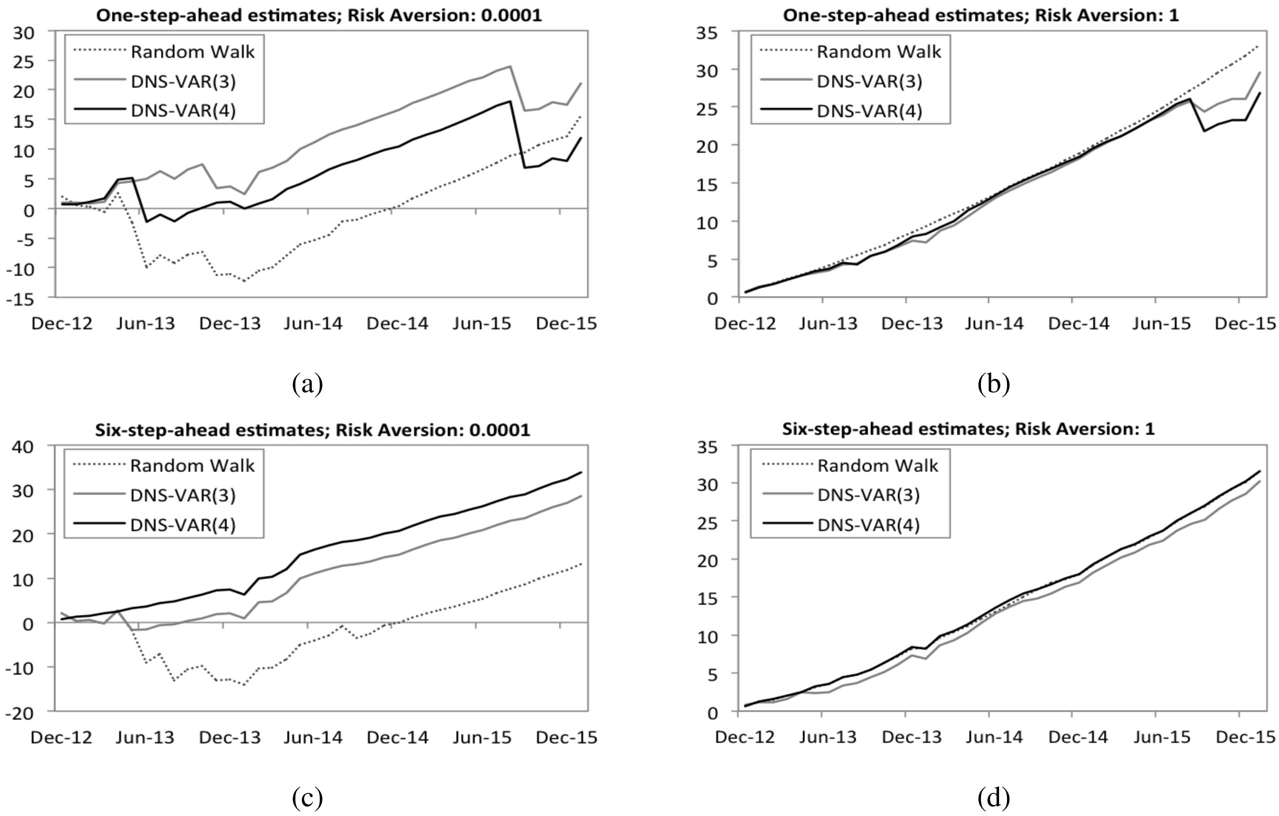


Figure 2: Cumulative net returns (in %): mean-variance portfolios with $\delta = 1 \times 10^{-4}$ and $\delta = 1$ for 1- and 6-step-ahead forecasts over the out-of-sample period.

It is also noteworthy the big drop in cumulative net returns for the one-step-ahead estimates in September 2015. At the end of August 2015, there is a deterioration of the Brazilian macroeconomic fundamentals due to the perception of a downturn in medium- and long-term fiscal scenario. Financial markets reacted with capital flight to safer investments and investors required higher premiums for holding Brazilian securities. Fig. 3 helps to visualize the scenario where short DI-futuro yields slightly rise while long-term yields suffer a large increase from July 2015, to September 2015, reflecting a large deterioration in long-term expectations about Brazilian macroeconomic foundations. Once investors with more risk-averse preferences tend to hold short-term maturities, they are not affected so much as the less risk-averse investors with $\delta = 1 \times 10^{-4}$ evidenced by panel (a) of Fig. 2.

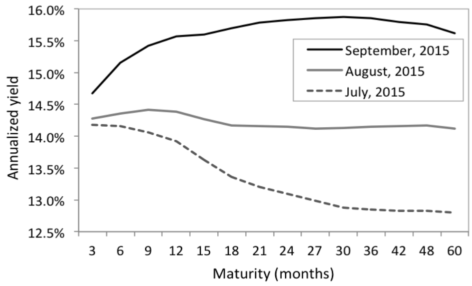


Figure 3: Observed yield curves from July 2015, to September 2015.

The link between Tables 3 and 4 is clear in the sense that better accuracy in yield curve forecasting leads to an improvement in terms of portfolio performance based on the mean-variance approach. In the general context, preferred one-month-ahead forecasted yields, which are generated by RW and *DNS-VAR(3)*, deliver better portfolio performance for optimizations with one-step-ahead estimates for DI-futuro returns. On the other hand, the *DNS-VAR(4)*^{2} specification, which consistently outperforms its competitors for forecast horizons longer than one month, reports better portfolio performance using six-step-ahead estimates. It is noteworthy that these findings highlight the relevance of good yield curve predictions to achieve better results in terms of portfolio performance. From this perspective, the incorporation of macroeconomic information into term structure models can play an important role to improve performance of fixed-income portfolios with biannual rebalancing.

5 Concluding remarks

The recent literature on yield curve forecasting suggests that the incorporation of a large macroeconomic dataset into term structure models improve forecast accuracy (De Pooter et al., 2010). Most of current studies test for statistical benefits from incorporating macroeconomic information, but little is known about the economic value of those forecasted yields. Besides testing for statistical improvement, this study uses a fixed-income portfolio analysis in order to assess the economic value of forecasted yields generated by yield curve models with macro factors extracted from a large macroeconomic dataset.

The out-of-sample forecast exercise support the evidence that a DNS yield curve model incorporating one macro factor, which summarizes broad information regarding primarily inflation expectations, outperforms the general DNS and RW models for forecast horizons longer than one month, specially at medium- and long-term maturities. This evidence indicates that an inflation factor is particularly useful to predict the Brazilian nominal yield curve dynamics.

The results for mean-variance portfolios show that better accuracy in yield curve forecasting leads to an improvement in terms of portfolio performance. Portfolio optimizations with six-step-ahead estimates for DI-futuro returns and biannual rebalancing report positive excess returns only for the DNS model which incorporates the inflation factor. Moreover, optimal portfolios with monthly rebalancing deliver higher return statistics for the general DNS model and RW, which forecasted yields outperform for the short forecast horizon scenario.

The overview indicates that good yield curve predictions are important to achieve economic gains from forecasted yields in terms of portfolio performance. It is clear that yield curve models with better forecast accuracy for short forecast horizons perform quite well for optimal mean-variance portfolios with one-step-ahead estimates for DI-futuro returns. In parallel, the DNS specification which incorporates an inflation factor outperforms in terms of portfolio performance with six-step-ahead estimates. Therefore, there is an economic and statistical gain from considering a large macroeconomic dataset to forecast the Brazilian yield curve dynamics, specially for longer forecast horizons.

References

Almeida, C. and A. Faria (2014). Forecasting the Brazilian term structure using macroeconomic factors. *Brazilian Review of Econometrics 34*(1), 45–77.

Ang, A. and M. Piazzesi (2003). A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables. *Journal of Monetary economics 50*(4), 745–787.

Bernanke, B. S. and J. Boivin (2003). Monetary policy in a data-rich environment. *Journal of Monetary Economics 50*(3), 525–546.

Bernanke, B. S., J. Boivin, and P. Eliasz (2005). Measuring the effects of monetary policy: A factor-augmented vector autoregressive (FAVAR) approach. *The Quarterly Journal of Economics 120*(1), 387–422.

Bolder, D. J. (2015). *Fixed-Income Portfolio Analytics: A Practical Guide to Implementing, Monitoring and Understanding Fixed-Income Portfolios*. Springer.

Caldeira, J. F., G. V. Moura, and A. A. P. Santos (2013). Measuring risk in fixed income portfolios using yield curve models. *Computational Economics 46*(1), 65–82.

Caldeira, J. F., G. V. Moura, and A. A. P. Santos (2016). Bond portfolio optimization using dynamic factor models. *Journal of Empirical Finance 37*, 128–158.

Caldeira, J. F., G. V. Moura, and M. Savino Portugal (2010). Efficient yield curve estimation and forecasting in brazil. *Revista Economia, January/April*.

Dai, Q. and K. J. Singleton (2000). Specification analysis of affine term structure models. *The Journal of Finance 55*(5), 1943–1978.

De Pooter, M., F. Ravazzolo, and D. J. Van Dijk (2010). Term structure forecasting using macro factors and forecast combination. *FRB International Finance Discussion Paper* (993).

Diebold, F. X. and C. Li (2006). Forecasting the term structure of government bond yields. *Journal of Econometrics 130*(2), 337–364.

Diebold, F. X. and G. D. Rudebusch (2013). *Yield Curve Modeling and Forecasting: The Dynamic Nelson-Siegel Approach*. Princeton University Press.

Diebold, F. X., G. D. Rudebusch, and S. B. Aruoba (2006). The macroeconomy and the yield curve: a dynamic latent factor approach. *Journal of Econometrics 131*(1), 309–338.

Duffee, G. R. (2002). Term premia and interest rate forecasts in affine models. *The Journal of Finance 57*(1), 405–443.

Durbin, J. and S. J. Koopman (2012). *Time series analysis by state space methods*. Number 38. Oxford University Press.

Favero, C. A., L. Niu, and L. Sala (2012). Term structure forecasting: No-arbitrage restrictions versus large information set. *Journal of Forecasting 31*(2), 124–156.

Giacomini, R. and H. White (2006). Tests of conditional predictive ability. *Econometrica 74*(6), 1545–1578.

Gürkaynak, R. S. and J. H. Wright (2012). Macroeconomics and the term structure. *Journal of Economic Literature 50*(2), 331–367.

Han, Y. (2006). Asset allocation with a high dimensional latent factor stochastic volatility model. *Review of Financial Studies 19*(1), 237–271.

Hördahl, P., O. Tristani, and D. Vestin (2006). A joint econometric model of macroeconomic and term-structure dynamics. *Journal of Econometrics 131*(1), 405–444.

Koopman, S. J. and M. Van der Wel (2013). Forecasting the US term structure of interest rates using a macroeconomic smooth dynamic factor model. *International Journal of Forecasting 29*(4), 676–694.

Korn, O. and C. Koziol (2006). Bond portfolio optimization: A risk-return approach. *Journal of fixed income 15*(4), 48–60.

Litterman, R. B. and J. Scheinkman (1991). Common factors affecting bond returns. *The Journal of Fixed Income 1*(1), 54–61.

Markowitz, H. (1952). Portfolio selection. *The Journal of Finance 7*(1), 77–91.

McCulloch, J. H. (1971). Measuring the term structure of interest rates. *The Journal of Business 44*(1), 19–31.

Moench, E. (2008). Forecasting the yield curve in a data-rich environment: A no-arbitrage factor-augmented var approach. *Journal of Econometrics 146*(1), 26–43.

Nelson, C. R. and A. F. Siegel (1987). Parsimonious mosdeling of yield curves. *Journal of Business*, 473–489.

Rossi, J. L. and M. D. Carvalho (2009). Identification of monetary policy shocks and its effects: Favar methodology for the Brazilian economy. *Brazilian Review of Econometrics 29*(2), 285–313.

Vieira, F., M. Fernandes, and F. Chague (2017). Forecasting the Brazilian yield curve using forward-looking variables. *International Journal of Forecasting 33*(1), 121–131.