# Exchange Rate Dynamics with Heterogeneous Expectations

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**Abstract**: As there is compelling empirical evidence that exchange rate expectations are persistently heterogeneous and formed mostly through boundedly rational mechanisms, we extend Dornbusch' (1976) model by incorporating heterogeneous exchange rate expectations. As in the Dornbusch model, but regardless of how boundedly rational behavior is specified, in the short run the exchange rate adjusts to a domestic monetary expansion by overshooting its long-run response. However, the overshooting (and hence the short-run volatility of the exchange rate) depends on the frequency distribution of strategies to form exchange rate expectations. The frequency distribution of exchange rate forecasting strategies follows an evolutionary dynamics. In line with the empirical evidence, both strategies to form exchange rate expectations survive in the long run. Yet perfect foresight by all agents is neither a necessary condition for realization of the fundamental values of the exchange rate and price level, nor an inevitable consequence of the achievement of these values. An important empirical implication of the model is then that expectations heterogeneity does matter for the volatility of the exchange rate.

**Keywords:** Exchange Rate Dynamics, Heterogeneous Expectations, Evolutionary Dynamics.

**Resumo**: Seguindo a evidência empírica de que as expectativas de taxa de câmbio são persistentemente heterogêneas e formadas principalmente por meio de mecanismos de racionalidade limitada, estendemos o modelo de Dornbusch (1976), incorporando expectativas cambiais heterogêneas. Como no modelo de Dornbusch, contudo, independentemente de como o comportamento limitadamente racional é especificado, no curto prazo, uma expansão monetária causará um *overshooting* da taxa de câmbio em relação ao seu valor de longo prazo. No entanto, a extensão do *overshooting* (e, portanto, a volatilidade de curto prazo da taxa de câmbio) depende da distribuição de frequência das estratégias de formação das expectativas cambiais. A distribuição das estratégias de previsão da taxa de câmbio segue uma dinâmica evolucionária. Em linha com a evidência empírica, as duas estratégias de formação de expectativas sobrevivem no longo prazo. Assim, a previsão perfeita por parte de todos os agentes não é nem uma condição necessária para a realização dos valores fundamentais da taxa de câmbio e do nível de preços, nem é uma consequência inevitável da realização destes valores fundamentais. Uma importante implicação empírica do modelo é que a heterogeneidade das expectativas tem impactos sobre a volatilidade da taxa de câmbio.

**Palavras-chave**: Dinâmica da taxa de câmbio, Expectativas heterogêneas, Dinâmica Evolucionária.

**JEL Classification:** C73, F31, F41

**Anpec Classification**: Área 7 – Economia Internacional.

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1. Introduction

As aptly pointed out by Rogoff (2002), Dornbusch's (1976) overshooting paper marks the birth of modern international macroeconomics. In a nutshell, Dornbusch's explanation for the excessive exchange rate volatility observed in the (then) recent transition from fixed to flexible exchange was that overshooting does not necessarily grows out of myopia or herd behavior in markets. Rather, exchange rate volatility is needed to temporarily equilibrate the system in response to monetary shocks, given that domestic prices adjust slowly. Besides, agents form exchange rate expectations that are consistent with the predictions of the model, and therefore have perfect foresight of the path of the economy following a monetary shock.

However, there is compelling empirical evidence from survey data that exchange rate expectations are persistently heterogeneous and formed mostly through boundedly rational mechanisms. Ito (1990), for example, analyses a panel of data collected bi-weekly by the Japan Center for International Finance on exchange rate expectations (yen/dollar) from forty four institutions, including banks, financial institutions, importers and exporters. The author finds strong evidence of heterogeneity amongst the market participants. While for the short term the hypothesis of rational expectations was rejected in some cases, in regard to six-month periods it was totally rejected. Besides, the paper finds evidence of lagged expectation formation. Allen and Taylor (1990), meanwhile, investigate the prevalence of technical analysts[1] (chartists) in the forex market of London. Using survey data, they find evidence that the technical analysis (chartist) is more usual in the short term. In fact, about 90% of the agents have reported to use some kind of technical analysis over short horizons, while for longer periods, there is evidence that chartists lose ground for the fundamental analysis. For periods of one year or more, 30% of the agents reported to be purely fundamentalists and 85% declared that the fundamentals were more important than technical analysis. Another result obtained by Allen and Taylor (1990) is that, even among chartists, there is expectational heterogeneity, with evidence of adaptive, regressive[2], extrapolative and bandwagon expectations. In fact, Taylor and Allen (1992), using data (questionnaire survey) from the Bank of England, find results that reinforce Allen and Taylor (1990).

Meanwhile, Cavaglia et al. (1998), using a survey data set of exchange rate expectations, corroborate the earlier finding in the literature that exchange rate forecasts are not rational and that agents do not use all available information efficiently. Although extrapolative and adaptive expectations formation mechanisms describe non-EMS exchange rate expectations to a certain extent, EMS exchange rates forecasts seem to follow long-run fundamentals more closely and would suggest that agents believe that EMS exchange rate expectations undershoot their long-run equilibrium values. Menkhoff (1998) studies the German exchange rate market through questionnaires sent to professionals from banks and fund management companies, and find evidence of the existence of two types of agents: agents who form fully rational expectations (arbitrageurs with fully-rational expectations) and agents who are subject to some kind of systematic bias (noise traders). The author also finds that even agents who use fundaments in a seemingly more apparent rational manner may be subject to some kind of bounded rationality. Chinn and Frankel (2002), despite not being explicitly interested

[1] The technical analysis consists in studying the series of asset prices in an attempt to formulate expectations either in an extrapolative way or not.

[2] Expectations of the type $e^{e}_{t+n} - e_t = \theta(\overline{e_t} - e_t)$.

in testing expectational heterogeneity, nonetheless find evidence of the presence of bounded rationality in the exchange market using survey data comprising 24 currencies against the U.S. dollar. The authors find a large variability in the expected exchange rate depreciation, as well as the existence of a time-varying risk premium. However, the authors do not find evidence of regressive expectations.

Motivated by this evidence and in the spirit of the contributions of Herbert Simon (to whom decision-makers have a limited capability to acquire, absorb, and process information), we extend Dornbusch' (1976) (representative-agent) open macroeconomy model by incorporating heterogeneous exchange rate expectations. There are two types of agents in the economy: *fully rational agents* who pay a cost to compute the correct (or *consistent*, to use Dornbusch's original expression) exchange rate forecast and *boundedly rational agents* who form "costless" regressive or static exchange rate forecasts. We should stress that what is costless about boundedly rational behavior is the *formation* of exchange rate expectations, and not *deviations* of the latter from the correct one. In fact, while fully rational agents pay an *ex ante* cost to avoid facing *ex post* losses resulting from forming an incorrect (or inconsistent) exchange rate expectation, boundedly rational agent do not pay an *ex ante* cost and are therefore subject to face *ex post* losses resulting from forming an incorrect (or inconsistent) exchange rate expectation. Differently from the original Dornbusch model, therefore, not all agents have perfect foresight and those who have perfect foresight have to pay a cost to become so. Hence, perfect foresight does not fall as manna from heaven. In this extension, however, we follow Dornbusch in assuming that output is fixed, and while the assets markets adjust fast enough (so that uncovered interest parity is always satisfied), the goods market adjust slowly.

Meanwhile, the frequency distribution of exchange rate forecasting strategies in the population is not treated as parametric, though. Rather, it follows an evolutionary dynamics driven by the relative performance of the available strategies. However, the model also features mutation as an exogenous disturbance in the evolutionary selection mechanism, leading some agents to choose an exchange rate forecasting strategy at random. This disturbance component is intended to capture the effect, for instance, of exogenous institutional factors, such as changes of administration in the monetary authority. Two other rationales for the existence of mutation are that an agent exits the economy with some (fixed) probability and is replaced with a new agent knowing nothing about the game, or that each agent simply "experiments" every once in a while with exogenously fixed probability.

In fact, an early extension in the same direction was set forth by Frankel and Froot (1986), who modeled the forex market as having three agents (with the frequency distribution of these agents treated as a parameter, though, unlike in the present paper): fundamentalists (who form exchange rate predictions based on the purchasing power parity), agents using time series (chartists) to predict the exchange rate depreciation, and portfolio managers who react to the two other types of agents, ultimately defining the exchange rate fluctuations. More recent studies incorporating heterogeneous exchange rate expectations (but not endogenously-varying as in the present paper) and bounded rationality in exchange rate models include Frenkel (1997), De Grauwe and Grimaldi (2006a; 2006b) and more recently De Grauwe and Markiewicz (2013).

Let us close this introduction of a preview of the main results derived along the way. As in Dornbusch's (representative-agent) model, but regardless of how boundedly rational behavior is specified, in the short run the exchange rate adjusts to a domestic monetary expansion by overshooting its long-run response. In this extension, though, the extent to the overshooting (and hence the short-run volatility of the exchange rate) depends on the frequency distribution of strategies to form exchange rate expectations. When boundedly rational agents form error-prone regressive expectations, the extent of the overshooting of the current exchange rate in response to a domestic monetary expansion varies positively (negatively) with the proportion of fully rational agents if boundedly rational agents' expected exchange depreciation overreacts (underreacts) to misalignments of the current exchange rate from the long-run exchange rate. Now, when boundedly rational agents form static exchange rate expectations (which means expecting the exchange rate not to change), the extent to the overshooting (and hence the short-run volatility of the exchange rate) unambiguously varies negatively with the proportion of fully rational agents.

Relevantly, as it is in accordance with the empirical evidence, both strategies to form exchange rate expectations survive in the long run. In fact, the model replicates, as an evolutionary equilibrium solution, the empirical evidence about the persistence of the heterogeneity in exchange rate expectations. Nonetheless, this does not preclude the exchange rate and the domestic price level from achieving their fundamental, long-run values. This result demonstrate that full rationality – a situation where all agents eventually adopt the perfect foresight strategy to form exchange rate expectations – is neither a necessary condition for realization of the fundamental values of the exchange rate and price level, nor an inevitable consequence of the economy's achievement of these values. Yet the equilibrium distribution of exchange rate forecasting strategies *does* affect the extent to which a domestic monetary expansion generates a short-run response of the exchange rate that overshoots its long-run response, as seen above. In fact, the result above about the long-run equilibrium configuration of the economy is robust to the consideration of the boundedly rational behavior as characterized by the formation of static exchange rate expectations. In fact, neither the long-run equilibrium configuration, nor its stability properties are affected by this change in how boundedly rational behavior is specified. However, the short-run volatility of the exchange rate is affected by the kind of boundedly rational behavior adopted by agents. Therefore, an important empirical implication of the model formulated in this paper is that the type and degree of heterogeneous expectations (as measured by the frequency distribution of strategies to forecast the exchange rate) does matter for the volatility of the current exchange rate.

The remainder of the paper is organized as follows. The next section lays out the structure of the basic model and analyses its behavior in the short run. While section 3 explores the behavior of the model in the long run, section 4 concludes.

## 2. Structure and behavior the model in the short run

In line with the empirical evidence reported in the preceding section, we extend Dornbusch's (1976) (representative-agent) open macroeconomic model to incorporate heterogeneous exchange rate expectations. Besides, in an innovative way, the frequency distribution of strategies to forecast the exchange rate is not treated as a parameter, but follows an evolutionary dynamics driven by the relative performance of the available strategies to perform these forecasts. Meanwhile, we follow Dornbusch's original model in assuming perfect capital mobility, a slow adjustment of goods markets relative to asset markets, and fixed real output at the full-employment level.[3]

We assume a small economy which therefore faces a given international rate of interest, $r^*$, with uncovered interest parity then implying that:

$$r = r^* + x \quad (1)$$

where $r$ is the domestic interest rate and $x$ is the expected rate of depreciation of the domestic currency (or the expected rate of increase of the domestic currency price of foreign exchange). As there is perfect capital mobility, (1) will hold continuously.

As regards expectations formation, the Dornbusch model distinguishes between the long-run exchange rate, to which the economy will ultimately converge, and the current exchange rate. Formally, it is assumed that:

$$x = \theta(\overline{e} - e) \quad (2)$$

where $e$ and $\overline{e}$ denote the logarithms of the current and long-run exchange rate, respectively, while $\theta > 0$ is the corresponding coefficient of adjustment. In the Dornbusch (representative-agent) model, it is later shown that the expectations formation mechanism given by (2) is actually consistent with perfect foresight.[4]

Unlike in the Dornbusch model, in this extension agents have heterogeneous expectations about the dynamics of the exchange rate. There are two types of agents: *fully rational agents* who pay a (fixed) cost to forecast the true (or *consistent*, to use Dornbusch's original expression) exchange rate dynamics and *boundedly rational agents* who either have regressive expectations (with imperfect foresight of the true dynamics of the exchange rate) or form a static exchange rate forecast (both boundedly rational strategies are costless). Therefore, perfect foresight does not fall as manna from heaven. In fact, while a fully rational agent pays an *ex ante* cost to avoid facing an *ex post* loss resulting from forming an incorrect (or inconsistent) forecast of the true dynamics of the exchange rate, a boundedly rational agent does not pay such an *ex ante* cost and is therefore subject to face an *ex post* loss resulting from forming an incorrect (or inconsistent) exchange rate expectation. Formally, while $k \in [0,1] \subset \mathbb{R}$ denotes the fraction of fully rational agents, $1-k \in [0,1] \subset \mathbb{R}$ denotes the fraction of boundedly rational agents. Both $k$ and (by extension) $1-k$ vary endogenously over time in an evolutionary manner that is described below. Therefore, instead of being given by (2) as in the original Dornbusch (representative-agent) model, expected depreciation is given by the following weighted average:

$$x = kx_f + (1-k)x_b \quad (3)$$

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[3] In the final section, Dornbusch (1976) explores the implications for exchange rate volatility of a variable output. We save for future research (to which we invite the reader to stay tuned) the further extension of the framework set forth in this paper to investigate the implications of this alternative assumption.

[4] The need to assume perfect foresight is justified by Dornbusch (1976) as follows: "Perhaps a remark about the perfect foresight path is in order here. Why should that path command our interest rather than being a mere *curiosum?* The reason is that it is the only expectational assumption that is not arbitrary (given the model) and that does not involve persistent prediction errors. The perfect foresight path is, obviously, the deterministic equivalent of rational expectations". (p. 1167, fn. 10) Clearly, Dornsbusch's plea for the need to assume that all agents have perfect foresight runs counter the empirical evidence reported in the preceding section.

where $x_f$ is the expected depreciation of fully rational agents and $x_b$ is the expected depreciation of boundedly rational agents. In line with the original Dornbusch model, the expected depreciation of fully rational agents is given by:

$$x_f = \theta_f(\bar{e} - e) \tag{4}$$

In a first specification of the boundedly rational behavior, meanwhile, the corresponding expected depreciation is given by:

$$x_b = \theta_b(\bar{e} - e) \tag{5}$$

where $\theta_i > 0$ and $\theta_b = \alpha\theta_f$, with $\alpha$ being a positive parameter which is different from, but close enough to, one. In this first specification, therefore, the long-run exchange rate is common knowledge and all agents expect the current exchange rate to converge to the corresponding long-run equilibrium value. However, even though all agents form expectations in such a regressive manner, only fully rational agents come to obtain (at an *ex ante* cost) perfect foresight about the actual path of the exchange rate as predicted by the model when expected depreciation varies across agents. Meanwhile, boundedly rational agents, having decided not to pay the corresponding *ex ante* cost to learn the actual path of the exchange rate, have imperfect foresight about it.

As in the Dornbusch model, we first solve for the path of the exchange rate predicted by the model (which in this extension depends on $\theta_b$ and $\theta_f$, and therefore on $k$) and then solve for the value of the adjustment coefficient $\theta_f$ that is consistent with the prediction of the model. Substituting (4) and (5) (along with $\theta_b = \alpha\theta_f$) into (3) (and re-arranging), we obtain:

$$x = A\theta_f\left(\bar{e} - e\right) \tag{6}$$

where $A(k) \equiv [k + \alpha(1-k)]$, with $A(k) > 0$ for all $k \in [0,1]$. However, note that $A(k)$ varies positively (negatively) with $k$ if $\alpha$ is smaller (greater) than one.

The domestic interest rate, $r$, is determined by the condition of equilibrium in the domestic money market, which is given by:

$$m = p + \phi y - \lambda r \tag{7}$$

where $m$, $p$ and $y$ denote the logs of the nominal supply of money, the price level and real output, respectively, while $\lambda$ and $\phi$ are positive parameters. Combining (1), (6) and (7), we get:

$$p - m = -\phi y + \lambda r^* + \lambda A\theta_f\left(\bar{e} - e\right) \tag{8}$$

which yields the relationship between the current exchange rate, the price level and the long-run exchange rate when the domestic money market clears and net asset returns are equalized. As we assume that the asset markets adjust fast enough, (8) will hold in any short run. Meanwhile, the long-run equilibrium configuration is characterized by the (average) expected depreciation being zero, which implies that (with an exogenously given nominal money supply) the long-run equilibrium price level is given by:

$$\bar{p} = m + (\lambda r^* - \phi y) \tag{9}$$

As it turns out, given the long-run values of the exchange rate and the price level, substituting (9) in (8) (and re-arranging) yields the following (negative) relationship between the current values of these variables:

$$e = \bar{e} - (1/\lambda A\theta_f)(p - \bar{p}) \tag{10}$$

Intuitively, a higher price level, by raising the domestic interest rate, leads to an incipient capital inflow that appreciates the current exchange rate to the extent required to generate an expectation of depreciation that exactly offsets the rise in the domestic interest rate. However, while in the Dornbusch model the exchange rate is assumed to adjust (much) faster than the price level, the present extension features the frequency distribution of strategies to form exchange rate expectations (given by $k$) as a further endogenous variable (and which is as slow moving as the price level).

Let us then move to the behavior of the domestic goods market. Recall that the real output is assumed to be fixed at the full-employment level, which implies that any excess demand for domestic output is inflationary as follows:

$$\dot{p} = \pi ln(D/Y) = \pi[u + \delta(e - p) + (\gamma - 1)y - \sigma r] \tag{11}$$

where $u$ represents autonomous components of demand, $\pi$, $\delta$, $\gamma$ and $\sigma$ are positive parameters, and the logarithm of the foreign price level has been normalized to zero. As the long-run equilibrium configuration is characterized by stationary values for the price level and the exchange rate (and hence by equalization of the interest rates), the long-run equilibrium exchange rate implied by (11) is given by:

$$\bar{e} = \bar{p} + (1/\delta)[\sigma r^* + (1-\gamma)y - u] \tag{12}$$

where $\bar{p}$ is given by (9). The inflation dynamics described in (11) can be simplified by using (12) and recalling from (1) and (6) that $r - r^* = A\theta_f(\bar{e} - e)$, which then yields:

$$\dot{p} = -\pi[(\delta + \sigma A\theta_f)/\lambda A\theta_f + \delta](p - \bar{p}) = -v(p - \bar{p}) \tag{13}$$

where:

$$v \equiv \pi[(\delta + \sigma A\theta_f)/\lambda A\theta_f + \delta] \tag{14}$$

Solving the differential equation (13), we get:

$$p(t) = \bar{p} - (p_0 - \bar{p})\exp(-vt) \tag{15}$$

Meanwhile, we can obtain the time path of the exchange rate by substituting (15) in (10) (and re-arranging), which yields:

$$e(t) = \bar{e} - (e_0 - \bar{e})\exp(-vt) \tag{16}$$

In the Dornbusch (representative-agent) model (in which, by assumption, $k = A = 1$), the adjustment coefficient given by (14) is parametric and positive, so that the implied prediction is that both the domestic price level and the exchange rate converge to the corresponding long-run equilibrium level. Now, in this extension with heterogeneous exchange rate expectations, even though $A(k) > 0$ for any $k \in [0,1]$, so that $v(k) > 0$ for any $k \in [0,1]$, the heterogeneity measure given by $k$ varies endogenously following an evolutionary dynamics driven by the relative performance of the strategies to form exchange rate expectations. Therefore, a natural question that arises is whether such an evolutionary dynamics thwarts the convergence of the domestic price level and the exchange rate to the corresponding long-run equilibrium levels. Moreover, as per (16) the time path of the exchange rate depends on the frequency distribution of strategies to forecast the exchange rate, the short-run response of the exchange rate to a (domestic) monetary shock (and hence the short-run volatility of the exchange rate) also depends on the heterogeneity measure given by $k$.

Now, for the expectation formation strategy described by (4) to correctly predict the actual dynamics of the exchange rate it must be the case that $\theta_f = v$. Therefore, the expectations coefficient of fully rational agents that corresponds to perfect foresight, $\tilde{\theta}$, can be obtained by solving the solving the following expression:

$$\theta_f = v \equiv \pi[(\delta + \sigma A\theta_f)/\lambda A\theta_f + \delta] \tag{17}$$

whose only positive (and hence stable) root of the implied quadratic equation is given by:

$$\tilde{\theta}(\lambda,\delta,\sigma,\pi,k) = \frac{\pi\left(\frac{\sigma}{\lambda}+\delta\right) + \left\{\left[\pi\left(\frac{\sigma}{\lambda}+\delta\right)\right]^2 + \frac{4\delta\pi}{\lambda A(k)}\right\}^{\frac{1}{2}}}{2} \tag{18}$$

As in the Dornbusch model, let us now explore the short-run implications of an increase in the nominal quantity of money that is expected to persist. In the Dornbusch model, which features a representative agent having perfect foresight, for given levels of the domestic price level (as the goods market adjusts slowly) and real output (assumed to be fixed), in the short run the exchange rate adjusts to a domestic monetary expansion by overshooting its long-run response (which is equal to such monetary expansion). In the present extension, we can use (8) (noting that $d\overline{e} = dm = d\overline{p}$) to obtain the following formal expression for the short-run response of the current exchange rate to a monetary expansion:

$$\frac{de}{dm} = 1 + \frac{1}{\lambda A(k)\tilde{\theta}(k)} > 1 \tag{19}$$

Hence, as in the Dornbusch (representative-agent) model, in the short run the exchange rate adjusts to a domestic monetary expansion by overshooting its long-run response. In this extension, though, the extent to the overshooting (and hence the short-run volatility of the exchange rate) also depends on the frequency distribution of strategies to form exchange rate expectations (which is as slow moving a variable as the domestic price level). In Appendix A1, it is shown that the extent of the overshooting of the current exchange rate varies positively (negatively) with the proportion of fully rational agents if $\alpha > 1$ ($\alpha < 1$).

In order to gain better intuition for the preceding result, we make some further assumptions about the parameters of the model. Recall that the parameter $\alpha$ was defined after (5) to measure the relationship between the expectations coefficient of boundedly rational agents, $\theta_b$, and a *given* expectations coefficient of (soon to become) fully rational agents, $\theta_f$, which is then made to correspond to the perfect foresight path, as given by (18). As our assumption that $\alpha$ is different from (but close enough to) one implies that $\theta_b$ is different from (but close enough to) $\theta_f$, we further assume that the parameters which determine $\tilde{\theta}$ in (18) are such that $\tilde{\theta}$ is different from (but close to)

$\theta_f$ for any $k \in [0,1]$ in an extent that allows us to take $\alpha > 1$ ($\alpha < 1$) as implying that $\theta_b > \tilde{\theta}$ ($\theta_b < \tilde{\theta}$). Therefore, $\alpha > 1$ ($\alpha < 1$) implies that, relatively to the perfect foresight path, boundedly rational agents' expected depreciation overreacts (underreacts) to any misalignment of the current exchange rate from the long-run exchange rate. Therefore, the extent of the overshooting of the current exchange rate in response to a domestic monetary expansion varies positively (negatively) with the proportion of fully rational agents if boundedly rational agents' expected depreciation overreacts (underreacts) to any misalignment of the current exchange rate from the long-run exchange rate.

We now consider an alternative specification of the boundedly rational behavior that conceives of it as being characterized by the formation of static exchange rate expectations. In this specification, the current exchange rate is assumed to be the best predictor of the future exchange rate, so that $x_b = 0$. As it turns out, average expected depreciation ceases to be given by (6) to be determined by:

$$x = k\theta_f \left(\bar{e} - e\right) \tag{20}$$

Combining (1), (7) and (20), we get the corresponding relationship between the current exchange rate, the price level and the long-run exchange rate when the domestic money market clears and net asset returns are equalized:

$$p - m = -\phi y + \lambda r^* + \lambda k\theta_f \left(\bar{e} - e\right) \tag{21}$$

As we assume that the asset markets adjust fast enough, (21) will hold in any short run. Meanwhile, we again obtain a negative relationship between the current values of these variables:

$$e = \bar{e} - (1/\lambda k\theta_f)(p - \bar{p}) \tag{22}$$

The dynamics of inflation is also derived as in the previous specification of boundedly rational behavior, which now yields:

$$\dot{p} = -\pi[(\delta + \sigma k\theta_f)/\lambda k\theta_f + \delta](p - \bar{p}) = -n(p - \bar{p}) \tag{23}$$

where:

$$n \equiv \pi[(\delta + \sigma k\theta_f)/\lambda k\theta_f + \delta] \tag{24}$$

Solving the differential equation (23), we get:

$$p(t) = \bar{p} - (p_0 - \bar{p})\exp(-nt) \tag{26}$$

Meanwhile, the time path of the exchange rate is given by:

$$e(t) = \bar{e} - (e_0 - \bar{e})\exp(-nt) \tag{27}$$

Now, for the expectation formation strategy described by (4) to correctly predict the actual dynamics of the exchange rate it must be the case that $\theta_f = n$. Therefore, the expectations coefficient of fully rational agents that corresponds to perfect foresight, $\hat{\theta}$, can be obtained by solving the solving the following expression:

$$\theta_f = n \equiv \pi[(\delta + \sigma k\theta_f)/\lambda k\theta_f + \delta] \tag{28}$$

whose only positive (and hence stable) root of the implied quadratic equation is given by:

$$\widehat{\theta}(\lambda,\delta,\sigma,\pi,k)=\frac{\pi\left(\frac{\sigma}{\lambda}+\delta\right)+\left\{\left[\pi\left(\frac{\sigma}{\lambda}+\delta\right)\right]^2+\frac{4\delta\pi}{\lambda k}\right\}^{\frac{1}{2}}}{2} \tag{29}$$

The formal expression for the short-run response of the current exchange rate to a monetary expansion is given by:

$$\frac{de}{dm}=1+\frac{1}{\lambda k\widehat{\theta}(k)}>1 \tag{30}$$

Therefore, as in the Dornbusch, in the short run the exchange rate adjusts to a domestic monetary expansion by overshooting its long-run response for any $k\in(0,1]$. In this extension, however, which features boundedly rational agents forming static exchange rate expectations, the extent to the overshooting (and hence the short-run volatility of the exchange rate) depends on the frequency distribution of strategies to form exchange rate expectations. In Appendix A2, it is shown that the extent of the overshooting of the current exchange rate varies negatively with the proportion of fully rational agents. In fact, having static exchange rate expectations is the most extreme form of underreacting (even if not realizing it) to any misalignment of the current exchange rate from the long-run exchange rate. As a result, it is not surprising that, unlike the previous specification of boundedly rational behavior, the extent of the overshooting of the current exchange rate unambiguously varies negatively with the proportion of fully rational agents for any $k\in(0,1]$. Note that for $k=0$, however, it follows that $\widehat{\theta}=\infty$, so that $(de/dm)=1$. In other words, if there is a stable long-run equilibrium configuration characterized by $k=0$, it follows that in an economy populated solely by boundedly rational agents with static exchange expectations, the short-run response of the exchange rate to a domestic monetary expansion does not involve either under- or overshooting. Nonetheless, as we will obtain in the next section, there is no long-run equilibrium solution characterized by $k=0$.

## 3. The behavior of the model in the long run

As intimated above, this extension of Dornbusch's model features the frequency distribution of strategies to forecast future exchange rates as endogenously time-varying (and as slow moving as the domestic price level), following an evolutionary dynamics. While a fully rational agent pays an *ex ante* (fixed) cost $z$ to avoid facing an *ex post* loss resulting from forming an incorrect (or inconsistent) exchange rate expectation, a boundedly rational agent does not pay an *ex ante* cost and is therefore subject to face an *ex post* loss resulting from the formation of an incorrect (or inconsistent) exchange rate expectation. Therefore, the loss of a fully rational agent is given by:

$$L_f=-z \tag{31}$$

Meanwhile, the loss of a boundedly rational agent is given by:

$$L_b=-\beta(x_f-x_b)^2 \tag{32}$$

As fully rational agents have perfect foresight, it follows that $\theta_f=\tilde{\theta}$, and substituting (18) in (4) yields $x_f=\tilde{\theta}(\overline{e}-e)$. Therefore, the exchange rate dynamics is described by:

$$\dot{e} = x_f = \tilde{\theta}(\bar{e} - e) \tag{33}$$

Substituting (5) and (33) in (32) and recalling that $\theta_b = \alpha\tilde{\theta}$, we get:

$$L_b(k) = -\beta\left[(1-\alpha)\tilde{\theta}(\bar{e} - e)\right]^2 \tag{34}$$

Following Weibull (1995), the frequency distribution of strategies to forecast future exchange rates is assumed to be given by a replicator dynamics:

$$\dot{k} = k(1-k)(L_f - L_b) \tag{35}$$

The logic of the replicator dynamics is that the frequency of a strategy in the population increases exactly when it has above-average payoff, $\bar{L}$, which in this case is given by $\bar{L} = kL_f + (1-k)L_b$.

We further assume that the evolutionary dynamics (35) driving the intertemporal behavior of the frequency distribution of strategies to form exchange rate expectations operates in the presence of a disturbance (noise) term, analogous to mutation in natural environments. In a biological setting, mutation is interpreted literally as consisting of random changes in genetic codes. In economic settings, as pointed out by Samuelson (1997, ch. 7), mutation refers to a situation in which a player refrains from comparing payoffs and changes strategy at random. Hence the present extension features mutation as an exogenous disturbance (noise) in the evolutionary mechanism leading some agents to choose an inflation foresight strategy at random. This disturbance component is intended to capture the effect, for instance, of exogenous institutional factors such as changes of administration in the monetary authority or other changes in the policy-making framework (which nonetheless do not involve an abandonment of the basic features of the policy architecture such as flexible exchange rates and perfect capital mobility). Or, as in Kandori, Mailath and Rob (1993), two other rationales for such a random choice are that an agent exits the economy with some (fixed) probability and is replaced with a new agent who knows nothing about (or is inexperienced in) the decision-making process, or that each agent simply "experiments" occasionally with exogenously fixed probability.

Drawing on Gale, Binmore and Samuelson (1995), mutation can be incorporated into the evolutionary dynamics (35) as follows. Let $\mu \in (0,1) \subset \mathbb{R}$ be the number (measure) of mutant agents that choose an exchange rate foresight strategy in a given revision period independently of the respective payoffs. Hence, there are $\mu(1-k)$ boundedly rational agents and $\mu k$ fully rational agents behaving as mutants. We assume that mutant agents choose either one or the other of the two exchange rate foresight strategies with equal probability. As a result, there are $\mu(1-k)/2$ boundedly rational mutant agents and $\mu k/2$ fully rational mutant agents changing foresight strategy. The net flow of mutant agents becoming fully rational agents in a given revision period, which can be either positive or negative, is then the following:

$$\mu(1-k)\frac{1}{2} - \mu k\frac{1}{2} = \mu\left(\frac{1}{2} - k\right) \tag{36}$$

Following Gale, Binmore and Samuelson (1995), this noise can be added to the evolutionary selection mechanism (35) to yield the following *noisy or perturbed satisficing evolutionary dynamics*:

$$\dot{k} = k(1-k)(L_f - L_b) + \mu\left(\frac{1}{2} - k\right) \quad (37)$$

Therefore, the dynamics of the economy is described by the three-dimensional system formed by (11), (33) and (37), whose long-run equilibrium configuration is given by $(e^*, p^*, k^*) = (\bar{e}, \bar{p}, k^* \in (0,1))$.[5] Relevantly, as it is quite in accordance with the empirical evidence, both strategies to form exchange rate expectations survive in the long run, with the resulting degree of heterogeneity being given by:

$$k^* = \frac{(z+\mu) - (z^2+\mu^2)^{\frac{1}{2}}}{2z} \quad (38)$$

Besides, the long-run (and hence persistent) heterogeneity in exchange rate expectations depends on its parametric determinants as follows:

$$\frac{\partial k^*}{\partial z} = \frac{\mu\left(\frac{\mu}{\sqrt{\mu^2+z^2}} - 1\right)}{2z^2} < 0 \quad (39)$$

$$\frac{\partial k^*}{\partial \mu} = \left(\frac{1}{2z}\right)\left(1 - \frac{\mu}{2(z^2+\mu^2)^{\frac{1}{2}}}\right) > 0 \quad (40)$$

Expectedly, the proportion of boundedly rational agents varies positively with the cost to acquire and process information to play the perfect foresight strategy, and negatively with the mutation rate.

Let us then conduct the corresponding stability analysis. The Jacobian matrix of the dynamic system evaluated around the equilibrium is given by:

$$J(\bar{p}, \bar{e}, k^*) = \begin{bmatrix} -\left(\pi\delta + \frac{\sigma}{\lambda}\right) & \pi\delta & 0 \\ 0 & -\tilde{\theta}(k^*) & 0 \\ 0 & 0 & (-z + 2k^*z - \mu) \end{bmatrix} \quad (41)$$

We can then set the following characteristic equation of the linearization around the equilibrium, in which $\varepsilon$ is an eigenvalue of the Jacobian matrix (41):

$$\left[(-z + 2k^*z - \mu) - \varepsilon\right]\left\{\left[\left(\pi\delta + \frac{\sigma}{\lambda}\right) + \varepsilon\right](\tilde{\theta} + \varepsilon)\right\} = 0$$

One of the eigenvalues can be obtained from the first part of the characteristic equation as follows:

$$(-z + 2k^*z - \mu) - \varepsilon = 0$$

which yields:

[5] As derived in Appendix A.3.

$$\varepsilon_1 = -\left(z^2 + \mu^2\right)^{\frac{1}{2}} < 0 \tag{42}$$

Meanwhile, we can the other two eigenvalues from the second part of the characteristic equation as follows:

$$\left[\left(\pi\delta + \frac{\sigma}{\lambda}\right) + \varepsilon\right]\left(\tilde{\theta} + \varepsilon\right) = 0 \tag{43}$$

which yields:

$$\varepsilon_2 = -\left(\pi\delta + \frac{\sigma}{\lambda}\right) \tag{44}$$

$$\varepsilon_3 = -\tilde{\theta} \tag{45}$$

Therefore, given that all these eigenvalues have strictly negative real parts, it follows that the equilibrium configuration given by $(\overline{e}, \overline{p}, k^* \in (0,1))$ is a local attractor. While the long-run equilibrium in the Dornbusch (representative-agent) model is saddle-point unstable (though it is shown that the economy will necessarily jump onto the stable arm of the new saddle path following a domestic monetary shock), in this extension the evolutionary dynamics with mutation therefore is able to stabilize the long-run equilibrium. Moreover, this result demonstrates that full rationality – a situation where all agents eventually adopt the perfect foresight strategy to form exchange rate expectations – is neither a necessary condition for realization of the fundamental values of the exchange rate and price level, nor an inevitable consequence of the economy's achievement of these values. Yet the equilibrium distribution of exchange rate forecasting strategies *does* affect the extent to which a domestic monetary expansion generates a short-run response of the exchange rate that overshoots its long-run response, as seen in the preceding section.

Interestingly, the above result is robust to the consideration of the alternative specification of the boundedly rational behavior as characterized by the formation of static exchange rate expectations (according to which the current exchange rate is taken to be the best predictor of the future exchange rate, so that the corresponding expected depreciation is zero). In fact, this alternative specification can be taken as a special case of the first specification by assuming that $\alpha = 0$. By checking (38)-(45), it can be easily verified that neither the long-run equilibrium configuration, nor its stability properties are affected by this change in the way boundedly rational behavior is specified. As seen in the preceding section, however, this alternative specification of boundedly rational behavior *does* affect the extent to which a domestic monetary expansion generates a short-run response of the exchange rate that overshoots its long-run response. In other words, the short-run volatility of the exchange rate is affected by the kind of boundedly rational behavior adopted by agents.

## 4. Conclusions

As there is compelling empirical evidence from survey data that exchange rate expectations are persistently heterogeneous and formed mostly through boundedly rational mechanisms, this paper extends Dornbusch' (1976) (representative-agent) open macroeconomy model by incorporating heterogeneous exchange rate expectations.

There are two types of agents in the economy: fully rational agents who pay a cost to compute the correct exchange rate forecast and boundedly rational agents who form "costless" regressive or static exchange rate forecasts. Differently from the original Dornbusch model, therefore, perfect foresight does not fall as manna from heaven.

Meanwhile, the frequency distribution of exchange rate forecasting strategies in the population follows an evolutionary dynamics driven by the relative performance of the available strategies and mutation. As in Dornbusch's (representative-agent) model, but regardless of how boundedly rational behavior is specified, in the short run the exchange rate adjusts to a domestic monetary expansion by overshooting its long-run response. In this extension, however, the extent to the overshooting (and hence the short-run volatility of the exchange rate) depends on the frequency distribution of strategies to form exchange rate expectations. When boundedly rational agents form error-prone regressive expectations, the overshooting varies positively (negatively) with the proportion of fully rational agents if boundedly rational agents' expected exchange depreciation overreacts (underreacts) to misalignments of the current exchange rate from the long-run exchange rate. Meanwhile, when boundedly rational agents form static exchange rate expectations, the overshooting unambiguously varies negatively with the proportion of fully rational agents.

In accordance with the empirical evidence, both strategies to form exchange rate expectations survive in the long run. In fact, the model replicates, as an evolutionary equilibrium solution, the empirical evidence about the persistence of the heterogeneity in exchange rate expectations. Yet this does not preclude the exchange rate and the domestic price level from achieving their fundamental values. While the long-run equilibrium in the Dornbusch (representative-agent) model is saddle-point unstable (though it is shown that the economy will necessarily jump onto the stable arm of the new saddle path following a domestic monetary shock), in this extension the evolutionary dynamics with mutation therefore is able to stabilize the long-run equilibrium. Moreover, this result demonstrates that full rationality – a situation where all agents eventually adopt the perfect foresight strategy to form exchange rate expectations – is neither a necessary condition for realization of the fundamental values of the exchange rate and price level, nor an inevitable consequence of the economy's achievement of these values. However, the equilibrium distribution of exchange rate forecasting strategies *does* affect the extent to which a domestic monetary expansion generates a short-run response of the exchange rate that overshoots its long-run response. Hence, an important empirical implication of the model formulated here is that the type and degree of heterogeneous expectations (as measured by the frequency distribution of strategies to forecast the exchange rate) does matter for the volatility of the current exchange rate.

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## Appendix A1

The extent of the overshooting is given by:

$$\frac{de}{dm} = 1 + \frac{2}{\lambda A(k)\left(\pi\left(\delta + \frac{\sigma}{\lambda}\right) + \sqrt{\pi^2\left(\delta + \frac{\sigma}{\lambda}\right)^2 + \frac{4\delta\pi}{\lambda A(k)}}\right)} \quad \text{(A1.1)}$$

Meanwhile, the extent of the overshooting varies with the frequency distribution of exchange rate forecasting strategies in the following way:

$$\frac{\partial\left(\frac{de}{dm}\right)}{\partial k} = \frac{(1-\alpha)(\mathrm{B}-C)}{D} \quad \text{(A1.2)}$$

where:

$$B = \frac{4\delta\pi}{\lambda^2 A^2(k)\sqrt{\pi^2\left(\delta + \frac{\sigma}{\lambda}\right)^2 + \frac{4\delta\pi}{\lambda A(k)}}} > 0 \quad \text{(A1.3)}$$

$$\mathrm{C} = 2\left(\pi\left(\delta + \frac{\sigma}{\lambda}\right) + \sqrt{\pi^2\left(\delta + \frac{\sigma}{\lambda}\right)^2 + \frac{4\delta\pi}{\lambda A(k)}}\right) > 0 \quad \text{(A1.4)}$$

$$D = \lambda A^2(k)\left(\pi\left(\delta + \frac{\sigma}{\lambda}\right) + \sqrt{\pi^2\left(\delta + \frac{\sigma}{\lambda}\right)^2 + \frac{4\delta\pi}{\lambda A(k)}}\right)^2 > 0 \quad \text{(A1.5)}$$

Note that $C > B$, given that:

$$4\delta\pi < 2\left(\pi\left(\delta + \frac{\sigma}{\lambda}\right) + \sqrt{\pi^2\left(\delta + \frac{\sigma}{\lambda}\right)^2 + \frac{4\delta\pi}{\lambda A(k)}}\right)\lambda^2 A^2(k)\sqrt{\pi^2\left(\delta + \frac{\sigma}{\lambda}\right)^2 + \frac{4\delta\pi}{\lambda A(k)}} \quad \text{(A1.6)}$$

Therefore:

$$\frac{\partial\left(\frac{de}{dm}\right)}{\partial k} = \frac{(1-\alpha)(\mathrm{B}-C)}{D}$$

## Appendix A2

The extent of the overshooting is given by:

$$\frac{de}{dm} = 1 + \frac{2}{\lambda k\left(\pi\left(\delta + \frac{\sigma}{\lambda}\right) + \sqrt{\pi^2\left(\delta + \frac{\sigma}{\lambda}\right)^2 + \frac{4\delta\pi}{\lambda k}}\right)} \quad \text{(A2.1)}$$

Let:

$$B' = \frac{4\delta\pi}{\lambda^2 k\sqrt{\pi^2\left(\delta + \frac{\sigma}{\lambda}\right)^2 + \frac{4\delta\pi}{\lambda k}}} \quad \text{(A2.2)}$$

$$C' = 2\left\{\pi\left(\delta + \frac{\sigma}{\lambda}\right) + \sqrt{\pi^2\left(\delta + \frac{\sigma}{\lambda}\right)^2 + \frac{4\delta\pi}{\lambda k}}\right\} \quad \text{(A2.3)}$$

$$D' = \lambda k^2\left\{\pi\left(\delta + \frac{\sigma}{\lambda}\right) + \sqrt{\pi^2\left(\delta + \frac{\sigma}{\lambda}\right)^2 + \frac{4\delta\pi}{\lambda k}}\right\}^2 \quad \text{(A2.4)}$$

Hence, the extent of the overshooting varies with the frequency distribution of exchange rate forecasting strategies in the following way:

$$\frac{\partial\left(\frac{de}{dm}\right)}{\partial k} = \frac{(B' - C')}{D} < 0 \quad \text{(A2.5)}$$

## Appendix A3

Recalling that $\tilde{\theta}$ (and $\hat{\theta}$) depends on $k$, the long-run equilibrium solution for $p$, $e$ and $k$ is obtained as follows:

$$0 = \pi\delta\left(e - \bar{e}\right) - \left(\pi\delta + \frac{\sigma}{\lambda}\right)\left(p - \bar{p}\right) \quad \text{(A3.1)}$$

$$0 = \tilde{\theta}\left(\bar{e} - e\right) \quad \text{(A3.2)}$$

$$0 = k\left(1 - k\right)\left[\beta[(1-\alpha)\tilde{\theta}(\bar{e} - e)]^2 - z\right] + \mu\left(\frac{1}{2} - k\right) \quad \text{(A3.3)}$$

It follows from (A.3.1) and (A.3.2) that $p = \bar{p}$ and $e = \bar{e}$. Substituting these solutions in (A3.3), we obtain:

$$k^2 z - \left(\mu + z\right)k + \frac{\mu}{2} = 0 \quad \text{(A3.4)}$$

There are two solutions for (A3.4):

$$k_1 = \frac{(\mu + z) + \sqrt{\mu^2 + z^2}}{2z} \tag{A3.5}$$

and:

$$k_2 = \frac{(\mu + z) - \sqrt{\mu^2 + z^2}}{2z} \tag{A3.6}$$

The solution given by (A3.5) is not economically meaningful, as it can assume values greater than one. This can be shown by taking the following limits:

$$\lim_{z \to 0} \frac{(z + \mu) + (z^2 + \mu^2)^{\frac{1}{2}}}{2z} = \lim_{z \to 0} \frac{\mu}{(z + \mu) - (z^2 + \mu^2)^{\frac{1}{2}}} = \infty \tag{A3.7}$$

$$\lim_{z \to \infty} \frac{(z + \mu) + (z^2 + \mu^2)^{\frac{1}{2}}}{2z} = \lim_{z \to 0} \frac{\left(1 + \frac{\mu}{z}\right) + \left(1 + \frac{\mu^2}{z^2}\right)^{\frac{1}{2}}}{2} = 1 \tag{A3.8}$$

$$\lim_{z \to 0} \frac{(z + \mu) - (z^2 + \mu^2)^{\frac{1}{2}}}{2z} = \lim_{z \to 0} \frac{\mu}{(z + \mu) + (z^2 + \mu^2)^{\frac{1}{2}}} = \frac{1}{2} \tag{A3.9}$$

$$\lim_{z \to \infty} \frac{(z + \mu) - (z^2 + \mu^2)^{\frac{1}{2}}}{2z} = \lim_{z \to 0} \frac{\mu}{(z + \mu) + (z^2 + \mu^2)^{\frac{1}{2}}} = 0 \tag{A3.10}$$