

On the Integral Representation of The Value Function*

Marcelo de Carvalho Griebeler[†]
PPGE/UFRGS

Jorge Paulo de Araújo[‡]
FCE/UFRGS

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Resumo: Neste trabalho propomos uma hipótese alternativa para a representação integral da função valor de Milgrom e Segal (2002). Ao invés de exigir que a função de utilidade possua derivada quase certamente, impomos que ela seja derivável em todo o seu domínio. A idéia é obter condições para aplicar o Teorema de Lebesgue, que fornece ao mesmo tempo uma função valor absolutamente contínua e sua representação integral. Nossa hipótese é tecnicamente mais forte do que a de Milgrom e Segal (2002), mas argumentamos que existe um substancial ganho de interpretação econômica ao adicioná-la. Enquanto é difícil interpretar continuidade absoluta em termos de preferências, a existência de derivada em todo ponto significa que todas as escolhas dos agentes – sobre qual tipo reportar – geram uma variação na sua utilidade.

Palavras-chave: função valor; desenho de mecanismos; diferenciabilidade.

Abstract: In this paper we propose an alternative assumption for the Milgrom and Segal (2002)'s integral representation of the value function. Instead of requiring that utility function has derivative almost everywhere, we impose that it has derivative in all its domain. The idea is to obtain conditions in order to apply the Lebesgue Theorem which provides at the same time an absolutely continuous value function and its integral representation. Our assumption is technically stronger than that of Milgrom and Segal (2002) but we argue that there is a substantial gain of economic interpretation in adding it. While it is difficult to interpret absolute continuity in terms of preferences, the existence of the derivative everywhere means that all agent's choices about what type to report generate a variation in his utility.

Keywords: value function; mechanism design; differentiability.

JEL: C60; C72; D86.

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[†]Address: Universidade Federal do Rio Grande do Sul, Faculdade de Ciências Econômicas, Av. João Pessoa, 52 sala 29J - 2º andar, Centro, Porto Alegre-RS, Brazil. ZIP code: 90040-000. E-mail: marcelogriebeler@gmail.com.

[‡]Address: Universidade Federal do Rio Grande do Sul, Faculdade de Ciências Econômicas, Av. João Pessoa, 52 sala 29J - 2º andar, Centro, Porto Alegre-RS, Brazil. ZIP code: 90040-000. E-mail: 00006283@ufrgs.br.

1 Introduction

Representing the value function in its integral form is a fundamental step in the process of solving mechanism design models with continuous type spaces. This step is part of an alternative approach first suggested by Mirrlees (1971) in his classic study about optimal taxation. Instead of using optimal control, in that paper is proposed a three-steps process of resolution. The idea is to eliminate one variable in the principal's optimization problem by using the indirect utility, provided by the value function. This is done by applying the Envelope Theorem (first step) and by writing the indirect utility in an integral form, in the second step. Thus, in the third step it is possible to replace this integral in the objective function and obtain a simplified expression to optimize.

Since the seminal paper of Mirrlees (1971) this approach has been used in studies applied to all fields of Economics. Examples are Mussa and Rosen (1978) in price discrimination, Baron and Myerson (1982) in Economics of Regulation and Maskin and Riley (1984) in non-linear monopoly pricing. More recently, Williams (1999) has provided some theoretical advances using that framework. All these papers have in common the standard assumption of differentiability of mechanisms. In fact, this requirement is crucial in the first step of the approach because the Envelope Theorem does not hold without differentiability¹. However, several examples in many different fields of Economics have showed that that assumption may not be satisfactory. Laffont and Tirole (1993) and Myerson (1991), for example, present discontinuous optimal mechanisms in Regulation Economics and bilateral trading, respectively.

Literature has tried to obtain the Envelope Theorem and the integral representation of the value function without differentiability. In this context, important studies are Krishna and Maenner (2001), Chung and Olszewski (2007) and Ely (2001). They focus their attention in developing general forms of the "payoff equivalence", with a particular case being the Revenue Equivalence Theorem. That literature has the advantage of allowing applications to multidimensional type spaces. However, by restricting the analysis to "payoff equivalence" its findings apply only to settings where the agent's utility is separable in the payment (or transfer) function. In other words, in order to obtain an integral representation of the value function, these authors require that the preferences are quasi-linear.

Milgrom and Segal (2002) are more general than the remainder of the literature because they obtain the two first steps of the Mirrlees's (1971) approach without differentiability and allowing that the utilities assume any functional form. In the other hand, the disadvantage of their study is the fact that it only applies to mechanism design models with a single agent. This limitation is overcome by Araújo and Griebeler (2009), that expand their Envelope Theorem to multidimensional type spaces by imposing an additional assumption. Through that new requirement those authors obtain a direct generalization of the integral representation of the value function. Nevertheless, according to Chung and Olszewski (2007), the assumption of absolute continuity assumed by these two papers to write the value function as an integral is not very elegant. In addition, it has no economic interpretation.

In this paper we propose an alternative assumption for the Milgrom and Segal (2002)'s integral representation of the value function. Instead of requiring that utility function has derivative almost everywhere, we impose that it has derivative in all its domain. The idea is to obtain conditions in order to apply the Lebesgue Theorem which provides at the same time an absolute continuous value function and its integral representation. Our assumption is technically stronger than that of Milgrom and Segal (2002) but we argue that there is a substantial gain of economic interpretation in adding it. While it is difficult to interpret absolute continuity in terms of preferences, the existence of the derivative everywhere means that all agent's choices - about what type to report - generate a variation in his utility.

Our paper is divided in three sections besides this brief introduction. The next one presents the integral representation of Milgrom and Segal (2002) and its direct generalization for multidimensional type spaces provided by Araújo and Griebeler (2009). We highlight the assumption of absolute continuity in this formulation and the criticisms regarded its little economic interpretation. In the third section we require the existence of the derivative everywhere in the agent's utility and obtain the same integral representation of the rest of the literature. In order to do so we use a modified version of the Vitali-Carathéodory Theorem and the Lebesgue Theorem. We also argue that our assumption has more economic interpretation than that of Milgrom and Segal (2002). Section four concludes.

2 The Integral Representation of Milgrom and Segal

We initially present the Milgrom and Segal's (2002) findings for one dimensional type space and later we analyse the generalization proposed by Araújo and Griebeler (2009). Consider the following framework. Let X be the choice set and let t be the relevant parameter such that $t \in I = [0, 1]$. Moreover, letting $f : X \times I \rightarrow \mathbb{R}$ denote the parameterized objective function, the value function V and the optimal choice correspondence (set-valued function) X^* are given by²:

¹For more details about the limitations caused by the non-differentiability in mechanism design models, see Araújo and Griebeler (2009).

²We adopt the original notation of the authors.

$$V(t) = \sup_{x \in X} f(x, t) \quad (1)$$

$$X^*(t) = \{x \in X : f(x, t) = V(t)\} \quad (2)$$

Observe that in mechanism design models $f(\cdot, t)$ may be seen as the utility function and t as the agent's type. It is important to note that the findings presented in this section also apply to any problem that has that same structure.

The result below is a generalization of the Envelope Theorem for arbitrary choice rules and it is necessary in the demonstration of the integral representation's theorem.

Theorem 1 *Take $t \in [0, 1]$ and $x^* \in X^*(t)$, and suppose that $f_t(x^*, t)$ exists. If $t > 0$ and V is left-hand differentiable at t , then $V'(t-) \leq f_t(x^*, t)$. If $t < 1$ and V is right-hand differentiable at t , then $V'(t+) \geq f_t(x^*, t)$. If $t \in [0, 1]$ and V is differentiable at t , then $V'(t) = f_t(x^*, t)$.*

Proof. See Milgrom and Segal (2002). ■

Theorem 1 is only useful when $V(\cdot)$ is well-behaved, such as differentiable, directionally differentiable, or absolutely continuous. Milgrom and Segal's (2002) second theorem provides conditions for the value function to have these desirable properties. In fact, since this result provides an absolutely continuous value function, then it has derivative almost everywhere and may be expressed in an integral form. Before we present it we introduce a lemma that will be important in the theorem's proof.

Lemma 2 *Assume the assumptions of Theorem 1, then for any $t_1 > t_2$ and $t_1, t_2 \in [0, 1]$,*

$$|V(t_1) - V(t_2)| \leq \sup_{x \in X} |f(x, t_1) - f(x, t_2)|. \quad (3)$$

Proof. See appendix. ■

Now we enunciate the Milgrom and Segal (2002)'s theorem which guarantees that the value function may be expressed in an integral form even when mechanism $x(\cdot)$ is not well-behaved.

Theorem 3 *Suppose that $f(x, \cdot)$ is absolutely continuous for all $x \in X$. Suppose also that there exists an integrable function $b : [0, 1] \rightarrow \mathbb{R}_+$ such that $|f_t(x(t), t)| \leq b(t)$ for all $x \in X$ and almost all $t \in [0, 1]$. Then $V(\cdot)$ is absolutely continuous. Suppose, in addition, that $f(x, \cdot)$ is differentiable for all $x \in X$, and that $X^* \neq \emptyset$ almost everywhere in $[0, 1]$. Then for any selection $x^*(t) \in X^*(t)$,*

$$V(t) = V(0) + \int_0^t f_t(x^*(s), s) ds. \quad (4)$$

Proof. Choose t' and t'' such that $t' \neq t''$. Then by the Lemma 2

$$\begin{aligned} |V(t'') - V(t')| &\leq \sup_{x \in X} |f(x, t'') - f(x, t')| \\ &\leq \sup_{x \in X} \left| \int_{t'}^{t''} f_t(x, s) ds \right| \\ &\leq \sup_{x \in X} \int_{t'}^{t''} |f_t(x, s)| ds \end{aligned} \quad (5)$$

because $f(x, \cdot)$ is absolutely continuous. Using the existence of the function b ,

$$|V(t'') - V(t')| \leq \sup_{x \in X} \int_{t'}^{t''} b(s) ds.$$

Taking the supremum in the expression above we may apply the sum and obtain

$$\sum_{i=1}^n \left| V(t_i'') - V(t_i') \right| \leq \sum_{i=1}^n \int_{t_i'}^{t_i''} b(s) ds,$$

what implies that $V(\cdot)$ is absolutely continuous.

In order to demonstrate the second part, observe that $V'(\cdot)$ exists, because $V(\cdot)$ is absolutely continuous. Thus integrating it we have

$$V(t) = V(0) + \int_0^t V'(s) ds,$$

but we know from Theorem 1 that $V'(t) = f_t(x^*(t), t)$. Therefore,

$$V(t) = V(0) + \int_0^t f_t(x^*(s), s) ds,$$

and the result is proved. ■

The necessity of imposing absolute continuity in the value function is stressed by Milgrom and Segal (2002) in commenting the fact that $V(\cdot)$ being differentiable almost everywhere does not imply that it equals the integral of the derivative. One possibility that is not kept out is the value function to be discontinuous. In general, differentiability almost everywhere is obtained in applications using monotonicity of the choice rule and the integral representation uses the specific functional forms for utility to be achieved. Myerson (1991), for example, uses a linear setting. Thus in the theorem's proof, what is important to note is that the assumption of absolute continuity is fundamental. In fact, it is not possible in this setting to obtain an upper bound in (5) without assuming that $f(\cdot; t)$ is absolutely continuous. Another point to highlight is the need of imposing the integrable function $b(\cdot)$, that makes the value function absolutely continuous and therefore to have derivative almost everywhere.

When we deal with t being a vector of types, the assumptions used in order to have an integral representation equivalent to (4) are the same, according to Araújo and Griebeler (2009). We present their main results below. Their first theorem is on differentiability of the value function and the form of its derivatives and generalizes the first theorem of Milgrom and Segal (2002) (Theorem 1 above). However, their statement is only related to differentiability, not focusing on the left-hand and right-hand derivatives. That result is necessary in expressing the value function in the integral form as we will see below.

Theorem 4 *Suppose that $f : X \times I \subseteq \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is a continuous function, with X being a compact subset of a Banach space $(\Omega, \|\cdot\|)$ and $I = \times_{i=1}^k [0, 1]$, and suppose also that $\nabla_t f : X \times I \subseteq \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is its continuous gradient. Then the value function*

$$V(t) = f(x^*(t), t) = \sup_{x \in X} f(x, t), \tag{6}$$

$V : I \subseteq \mathbb{R}^m \rightarrow \mathbb{R}$, is differentiable and $\nabla_t V(t) = \nabla_t f(x^(t), t)$, where $x^*(t) \in \arg \max(t)$.*

Proof. See Araújo and Griebeler (2009). ■

The theorem of Araújo and Griebeler (2009) has stronger assumptions than those of Milgrom and Segal (2002). For example, compactness of X and continuity of $\nabla_t f(x(t))$ are not necessary to obtain the result for unidimensional type spaces. There is a loss of generality when we expand the results for multi-agent settings, in the sense of not allowing an arbitrary structure for the choice set. However, that study has the advantage of continuing allowing that choice rule $x(\cdot)$ is arbitrary, like discontinuous, for example. The inclusion of the compactness of X is justified by the difficulty of this assumption to be violated in applications.

Before the integral representation for multidimensional type spaces, we present another lemma used in its proof. Notice below that Theorem 6 is a direct generalization of Milgrom and Segal's (2002) findings.

Lemma 5 *Assume the assumptions of Theorem 2, then for any vectors $t^1, t^2 \in I = \times_{i=1}^k [0, 1]$, $t^1 \neq t^2$,*

$$|V(t^1) - V(t^2)| \leq \sup_{x \in X} |f(x, t^1) - f(x, t^2)|. \tag{7}$$

Proof. See Araújo and Griebeler (2009). ■

Theorem 6 Suppose that $f(x, \cdot)$ is absolutely continuous for all $x \in X$. Suppose also that there exists an integrable function $b : I = \times_{i=1}^k (0, 1) \rightarrow \mathbb{R}_+$ such that $\left| \frac{\partial f(x, t_1, t_2, \dots, t_k)}{\partial t_i} \right| \leq b(t_1, t_2, \dots, t_k)$ for all $x \in X$, all $i = 1, \dots, k$, and almost all $(t_1, t_2, \dots, t_k) \in I$. Then V is absolutely continuous. Suppose, in addition, that $f(x, \cdot)$ is differentiable for all $x \in X$, and that $X^*(t_1, t_2, \dots, t_k) \neq \emptyset$ almost everywhere on I . Then for any selection $x^*(t_1, t_2, \dots, t_k) \in X^*(t_1, t_2, \dots, t_k)$,

$$V(t_1, t_2, \dots, t_k) = V(0, \dots, 0) + \sum_{i=0}^k \int_0^{t_i} \frac{\partial f}{\partial t_i}(x, s) ds. \quad (8)$$

Proof. Taking $t^1 = (t_1, \dots, t_j'', \dots, t_k)$ and $t^2 = (t_1, \dots, t_j', \dots, t_k)$, by Lemma 5 we know that

$$\begin{aligned} \left| V(t_1, \dots, t_j'', \dots, t_k) - V(t_1, \dots, t_j', \dots, t_k) \right| &\leq \sup_{x \in X} \left| f(x, t_1, \dots, t_j', \dots, t_k) - f(x, t_1, \dots, t_j'', \dots, t_k) \right| \\ &\leq \sup_{x \in X} \left| \int_{t_j'}^{t_j''} \frac{\partial f}{\partial t_j}(x, s) ds \right| \\ &\leq \sup_{x \in X} \int_{t_j'}^{t_j''} \left| \frac{\partial f}{\partial t_j}(x, s) \right| ds \\ &\leq \int_{t_j'}^{t_j''} b(s) ds. \end{aligned}$$

This implies that V is absolutely continuous. Therefore, $\frac{\partial V}{\partial t_j}$ exists almost everywhere, and we might write

$$V(t_1, \dots, t_j, \dots, t_k) = V(t_1, \dots, 0, \dots, t_k) + \int_0^{t_j} \frac{\partial V}{\partial t_j}(x, s) ds,$$

where 0 is in the j -th component of the vector t in the right side of the equation. It is possible to do the same for each component t_j . To see this, suppose that $j = k$. Thus, we have

$$\begin{aligned} V(t_1, t_2, \dots, t_k) &= V(t_1, t_2, \dots, t_{k-1}, 0) + \int_0^{t_k} \frac{\partial V}{\partial t_k}(x, s) ds \\ &= V(t_1, t_2, \dots, t_{k-2}, 0, 0) + \int_0^{t_{k-1}} \frac{\partial V}{\partial t_{k-1}}(x, s) ds + \int_0^{t_k} \frac{\partial V}{\partial t_k}(x, s) ds \\ &\quad \vdots \\ &= V(t_1, t_2, \dots, t_k) = V(0, \dots, 0) + \sum_{i=0}^k \int_0^{t_i} \frac{\partial V}{\partial t_i}(x, s) ds. \end{aligned}$$

If $f(x, t)$ is differentiable, $\frac{\partial V}{\partial t_i}(x, s) ds$ is given by Theorem 4 wherever it exists, then

$$V(t_1, t_2, \dots, t_k) = V(0, \dots, 0) + \sum_{i=0}^k \int_0^{t_i} \frac{\partial f}{\partial t_i}(x, s) ds.$$

■

As we mentioned above, in this second result it is not necessary to make more assumptions than those made by Milgrom and Segal (2002) in Theorem 3. Following the earlier theorem's fashion, it is only imposed structure in $f(\cdot)$, so Araújo and Griebeler (2009) continue to allow $x(\cdot)$ to be arbitrary. Moreover, since it is a direct generalization its limitations are the same of Theorem 3. The main limitation is the assumption of absolute continuity of $f(\cdot; t)$. In fact, part of literature considers this requirement not elegant. Chung and Olszewski (2007), for example, in their application to Revenue Equivalence Principle, stated that explicitly. Moreover, when we think about other applications, absolute continuity has no economic meaning. In any mechanism design model that we consider, $f(\cdot; t)$ represents the agent's utility function. Thus it should be possible to obtain some interpretation of that assumption in terms of preferences.

Nevertheless, the fact of the utility function is absolutely continuous means nothing, economically speaking.

What one may conclude of the inclusion of that assumption comes from some indirect results. For instance, since every absolutely continuous function is of bounded variation, we may conclude that so is the agent's utility. This conclusion has more economic meaning since a model with an agent that has an utility of unbounded variation (his utility may achieve infinity) makes no sense. In addition, absolute continuity implies that derivative exists almost everywhere. In our context, it means that agent's marginal utility with respect to his type, $\frac{\partial f}{\partial t}$, does not exist only in a set with null measure. We deal with these limitations in the next section and propose an alternative with more economic interpretation in order to overcome it.

3 Main Results

As we showed in section 2, existence almost everywhere of the utility function's derivative is a consequence of the assumption of absolute continuity. Here we make a technically stronger requirement, namely we require that agent's utility function has derivative everywhere. There are two reason to include it. The first one is that with this additional assumption we have more economic interpretation. Instead of existing almost everywhere, the agent's marginal utility with respect to his type exists everywhere. It is very plausible that for any type chosen by the agent his utility varies locally.

The other reason is that we are only worried in letting the mechanism be arbitrary, following the related literature. Thus, in that sense, it is not a limitation impose some structure in the preferences. In fact, in applications it is difficult to see an example of utility functions that violate the existence of the derivative everywhere. Therefore we do so below. However, before that we need some auxiliaries results³.

Lemma 7 *Let $f : X \rightarrow [0, +\infty]$ be a measurable function, then there exists a sequence (φ_n) of simple functions such that $\varphi_n(x) \uparrow f(x)$.*

Proof. See appendix. ■

The theorem below is a modified version of a result known as Vitali-Carathéodory Theorem.

Theorem 8 *Let $f : X \rightarrow [-\infty, +\infty]$ be a measurable function. Then for all $\varepsilon > 0$ there exist u and v such that $u \leq f \leq v$ and*

$$\int_X (f - u) d\mu < \varepsilon \quad (9)$$

and

$$\int_X (v - f) d\mu < \varepsilon, \quad (10)$$

where u is upper hemicontinuous and v is lower hemicontinuous.

Proof. See appendix. ■

Now we enunciate and demonstrate the result that garantees the integral representation (4) using the existence of the derivative everywhere of f .

Theorem 9 *Let $f : [a, b] \rightarrow \mathbb{R}$ be a function with derivative in all its domain and let $f' \in L^1([a, b])$, then*

$$\int_a^b f'(t) dt = f(b) - f(a) \quad (11)$$

and f is absolutely continuous.

Proof. We know from Theorem 8 that, for all $\varepsilon > 0$, there exists an upper hemicontinuous function v such that $f' \leq v$. Choosing a constant $c > 0$ in a suitable way, there exists a lower hemicontinuous function $g = v + c$ such that $f' < g$ and $\int_a^b g(t) dt < \int_a^b f'(t) dt + \varepsilon$.

For all $\eta > 0$ we define

$$F_\eta(x) = \int_a^x g(t) dt - f(x) + f(a) + \eta(x - a), \quad (12)$$

³Mathematical concepts used in this section may be found in Royden (1968) and Rudin (1986).

with $a \leq x \leq b$.

Let $x \in [a, b]$. Since $f'(x) < g(x)$ and $g(\cdot)$ is lower hemicontinuous, then there exists $(x, x + \delta)$ such that $t \in (x, x + \delta)$ implies that $f'(x) < g(t)$ and

$$\frac{f(t) - f(x)}{t - x} < f'(x) + \eta, \quad (13)$$

because $f(\cdot)$ is differentiable at x .

In addition, we have

$$\begin{aligned} F_\eta(t) - F_\eta(x) &= \int_x^t g(s)ds - (f(t) - f(x)) + \eta(t - x) \\ &> f'(x)(t - x) - (f'(x) + \eta)(t - x) + \eta(t - x) \\ &= 0. \end{aligned}$$

Since $F_\eta(a) = 0$ and $F_\eta(\cdot)$ is continuous, we may define $x^* = \sup_{\substack{x \in [a, b] \\ F_\eta(x) = 0}} x$. If $x^* < b$, then by the calculation above we have $F_\eta(t) > 0$ for $t \in (x, b]$. Therefore, $F_\eta(b) \geq 0$. Using the definition (12) we have

$$F_\eta(b) = \int_a^b g(t)dt - f(b) + f(a) + \eta(b - a) \geq 0$$

and as $\eta \rightarrow 0^+$

$$\begin{aligned} \int_a^b g(t)dt &\geq f(b) - f(a) \\ \int_a^b f'(t)dt + \varepsilon &> f(b) - f(a), \end{aligned}$$

where we use (13). As $\varepsilon \rightarrow 0^+$,

$$\int_a^b f'(t)dt \geq f(b) - f(a).$$

In repeating the process to $-f'$, we obtain

$$\begin{aligned} \int_a^b -f'(t)dt &\geq (-f(b)) - (-f(a)) \\ \int_a^b f'(t)dt &\leq f(b) - f(a). \end{aligned}$$

Moreover, by the Lebesgue Theorem, if f' exists almost everywhere and it is integrable and $\int_a^b f'(t)dt = f(b) - f(a)$ holds, then f is absolutely continuous. ■

Observe that whole demonstration was made for unidimensional case, but the generalization is direct if we require existence of $\frac{\partial f}{\partial t_i}$ in all utility function's domain (in this case the type spaces becomes $I = \times_{i=1}^k [a, b]$) and that $\frac{\partial f}{\partial t_i} \in L^1([a, b])$ for every $i = 1, \dots, k$. With these assumptions part of the procedure of Theorem 6's proof would be the same and we would have (8) directly. Also, notice that differentiability is the only one additional requirement in the theorem. In fact, in requiring that $f' \in L^1([a, b])$ we do not impose any other additional assumption, since it is equivalent to the integrable function $b(\cdot)$ proposed by Milgom and Segal (2002) in Theorem 3.

4 Concluding Remarks

Since Mirrlees (1971) the importance of representing the value function in mechanism design models as an integral has been highlighted. Several applications have used this result in order to simplify the solving process. Examples are the seminal papers of Mussa and Rosen (1978), Baron and Myerson (1982) and Maskin and Riley (1984). However, all this literature assumes differentiable mechanisms. As the applications have been developed, examples of discontinuous mechanisms have raised and that standard framework has become unsatisfactory. In recent years, studies like Milgrom and Segal (2002) has overcome this limitation by adding the assumption of absolute continuity of the utility function. Yet, part of literature considers this assumption not elegant and with no economic meaning.

In this paper we propose an alternative assumption for the Milgrom and Segal (2002)'s integral representation of the value function. Instead of requiring that utility function has derivative almost everywhere, we impose that it has derivative in all its domain. The idea is to obtain conditions in order to apply the Lebesgue Theorem which provides at the same time an absolutely continuous value function and its integral representation. Our assumption is technically stronger than that of Milgrom and Segal (2002) but we argue that there is a substantial gain of economic interpretation in adding it. While it is difficult to interpret absolute continuity in terms of preferences, the existence of the derivative everywhere means that all agent's choices - about what type to report - generate a variation in his utility.

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5 Appendix

Proof. (Lemma 2) Using the definition of $V(\cdot)$,

$$\begin{aligned} V(t_1) - V(t_2) &= f(x(t_1), t_1) - f(x(t_2), t_2) \leq f(x(t_1), t_1) - f(x(t_1), t_2) \\ &\leq f(x(t_1), t_1) - f(x(t_1), t_2) \\ &\leq \sup_{x \in X} |f(x, t_1) - f(x, t_2)|. \end{aligned}$$

Similarly,

$$\begin{aligned}
V(t_2) - V(t_1) &= f(x(t_2), t_2) - f(x(t_1), t_1) \leq f(x(t_2), t_2) - f(x(t_2), t_1) \\
&\leq f(x(t_2), t_2) - f(x(t_2), t_1) \\
&\leq \sup_{x \in X} |f(x, t_2) - f(x, t_1)|.
\end{aligned}$$

Since $|f(x, t_1) - f(x, t_2)| = |f(x, t_2) - f(x, t_1)|$ the same must hold when we take the supremum in both sides, $\sup_{x \in X} |f(x, t_1) - f(x, t_2)| = \sup_{x \in X} |f(x, t_2) - f(x, t_1)|$. Thus,

$$|V(t_1) - V(t_2)| \leq \sup_{x \in X} |f(x, t_1) - f(x, t_2)|.$$

■

Proof. (Lemma 7) Let's start by defining the following sets:

$$\begin{aligned}
E_1 &= \{x \in X | f(x) \geq 1\} \\
E_2 &= \left\{ x \in X | f(x) \geq \frac{1}{2} + \chi_{E_1}(x) \right\} \\
&\vdots \\
E_k &= \left\{ x \in X | f(x) \geq \frac{1}{k} + \sum_{j=1}^{k-1} \chi_{E_j}(x) \right\},
\end{aligned}$$

for $k \geq 2$ and where $\chi_{E_j}(\cdot)$ is the characteristic function of the set E_j . Notice that since E_1 is measurable so is E_2 , because we may write $f(x) - \chi_{E_1}(x) \geq 1$. Doing the same reasoning we obtain that E_k is measurable for every k .

Now we define the function $\varphi_k(x) = \sum_{j=1}^k \frac{1}{j} \chi_{E_j}(x)$. Clearly, $\varphi_k(\cdot)$ is monotonically nonincreasing because $\varphi_k(x) = \sum_{j=1}^k \frac{1}{j} \chi_{E_j}(x) \leq \varphi_{k+1}(x) = \sum_{j=1}^{k+1} \frac{1}{j} \chi_{E_j}(x)$. We must demonstrate that $\varphi_k(x) \leq f(x)$.

Suppose by absurd that $\varphi_k(x) > f(x)$. Thus, there exist k_0 and x' such that $0 \leq f(x') < \varphi_{k_0}(x')$, implying $\varphi_{k_0}(x') = \sum_{j=1}^{k_0} \frac{1}{j} \chi_{E_j}(x') > 0$. Then, $x' \in E_j$ for some $1 \leq j \leq k_0$. Set $k' = \max_{x' \in E_j} j$. If $k' = 1$ so $f(x') \geq 1 = \varphi_{E_1}(x')$.

Similarly, if $k \geq 2$ we have $x' \in E_{k'}$, such that

$$f(x') \geq \frac{1}{k'} + \sum_{j=1}^{k'-1} \frac{1}{j} \chi_{E_j}(x') = \sum_{j=1}^{k'} \frac{1}{j} \chi_{E_j}(x') = \varphi_{k'}(x').$$

Therefore we have $f(x') \geq \varphi_{k'}(x')$ e $k' < k_0$, because by assumption $f(x') < \varphi_{k_0}(x')$. Using the definitions,

$$\begin{aligned}
0 &\leq f(x') - \varphi_{k'}(x') = f(x') - \sum_{j=1}^{k'} \frac{1}{j} \chi_{E_j}(x') \\
&< \sum_{j=1}^{k_0} \frac{1}{j} \chi_{E_j}(x') - \sum_{j=1}^{k'} \frac{1}{j} \chi_{E_j}(x') \\
&= \sum_{j=k'+1}^{k_0} \frac{1}{j} \chi_{E_j}(x').
\end{aligned}$$

However the fact that $0 < \sum_{j=k'+1}^{k_0} \frac{1}{j} \chi_{E_j}(x')$ implies that there exists a $k'' > k'$ such that $x' \in E_{k''}$, what contradicts our initial assumption that k' is the maximum. Thus, $\varphi_k(x) \leq f(x)$.

We still need to show that $\varphi_n(x) \uparrow f(x)$. In the case of $f(x) = +\infty$, $x \in E_j$ for every j , such that $\varphi_n(x) =$

$1 + \frac{1}{2} + \dots + \frac{1}{n} \rightarrow \infty$ as $n \rightarrow \infty$. When $f(x) < +\infty$, there is no N such that $x \in E_j$, for every $j \geq N$, because in the case of existence, $\varphi_j(x) = \frac{1}{N} + \frac{1}{N+1} + \dots + \frac{1}{j} \rightarrow \infty$. Then, there exists k_m such that $x \notin E_j$ for $j \geq k_m$. Therefore,

$$\begin{aligned} f(x) &< \frac{1}{k_m} + \sum_{j=1}^{k_m-1} \frac{1}{j} \chi_{E_j}(x) \\ \Rightarrow 0 &\leq f(x) - \varphi_{k_m-1}(x) < \frac{1}{k_m} \\ \Rightarrow \varphi_{k_m-1}(x) &\rightarrow f(x). \end{aligned}$$

Since $\varphi_n(x)$ is monotonic, we have $\varphi_n(x) \uparrow f(x)$. ■

Proof. (Theorem 8) We first demonstrate in the case of $f(x) \geq 0$. Using Lemma 7, we may write $f(x) = \sum_{j=1}^{\infty} \frac{1}{j} \chi_{E_j}(x)$. Since the sets E_j are measurable, for all $\varepsilon > 0$, there exists an open U_j such that $E_j \subseteq U_j$ and $\mu(U_j \setminus E_j) < \frac{\varepsilon}{2^j}$. Thus, we may define

$$v(x) = \sum_{j=1}^{\infty} \frac{1}{j} \chi_{U_j}(x)$$

and verify that $f(x) = \sum_{j=1}^{\infty} \frac{1}{j} \chi_{E_j}(x) \leq \sum_{j=1}^{\infty} \frac{1}{j} \chi_{U_j}(x) = v(x) = v$.

Since that each U_j is open then $\frac{1}{j} \chi_{U_j}(x)$ and $v(x)$ are lower hemicontinuous, because a series of lower hemicontinuous functions is also lower hemicontinuous.

Now we may obtain (10). Notice that

$$\begin{aligned} \int_X (v - f) d\mu &= \sum_{j=1}^{\infty} \frac{1}{j} \mu(U_j \setminus E_j) \\ &= \sum_{j=1}^{\infty} \frac{1}{j} \frac{\varepsilon}{2^j} < \varepsilon. \end{aligned}$$

Now let's show (9). For each E_j there also exists a closed K_j such that $K_j \subseteq E_j$ and $\mu(E_j \setminus K_j) < \frac{\varepsilon}{2^j}$. Proceeding in the same way above,

$$\sum_{j=1}^{\infty} \frac{1}{j} \chi_{K_j}(x) \leq \sum_{j=1}^{\infty} \frac{1}{j} \chi_{E_j}(x) = f(x)$$

and integrating

$$\int_X \left(f - \sum_{j=1}^{\infty} \frac{1}{j} \chi_{K_j}(x) \right) d\mu = \sum_{j=1}^{\infty} \frac{1}{j} \mu(E_j \setminus K_j) < \varepsilon.$$

Notice that it is possible to choose a $N < \infty$ such that

$$\int_X \left(f - \sum_{j=1}^N \frac{1}{j} \chi_{K_j}(x) \right) d\mu < \varepsilon.$$

Defining

$$u(x) = \sum_{j=1}^N \frac{1}{j} \chi_{K_j}(x),$$

we have that $u(x)$ is upper hemicontinuous because K_j is closed. Thus,

$$\int_X (f - u) d\mu < \varepsilon.$$

In the general case, with $-\infty \leq f(x) \leq +\infty$, we separate f in its positive and negative parts, $f = f^+ - f^-$. Since both parts are nonnegative we must have $u^+ \leq f^+ \leq v^+$ and $u^- \leq f^- \leq v^-$, where the functions u 's are upper hemicontinuous and the v 's are lower hemicontinuous. By changing the signal of the functions related to negative parts,

we have $-v^- \leq -f^- \leq -u^-$, where now $-u^-$ is lower hemicontinuous and $-v^-$ is upper hemicontinuous. Then,

$$\hat{u} = u^+ - v^- \leq f \leq v^+ - u^- = \hat{v}.$$

Moreover,

$$\begin{aligned} \int_{\bar{X}} (f - \hat{u}) d\mu &= \int_{\bar{X}} ((f^+ - f^-) - (u^+ - v^-)) d\mu \\ &= \int_{\bar{X}} (f^+ - u^+) d\mu + \int_{\bar{X}} (v^- - f^-) d\mu \\ &< 2\varepsilon \end{aligned}$$

and

$$\begin{aligned} \int_{\bar{X}} (\hat{v} - f) d\mu &= \int_{\bar{X}} ((v^+ - u^-) - (f^+ - f^-)) d\mu \\ &= \int_{\bar{X}} (v^+ - f^+) d\mu + \int_{\bar{X}} (f^- - u^-) d\mu \\ &< 2\varepsilon. \end{aligned}$$

■