GIVING FLEXIBILITY TO THE NELSON-SIEGEL CLASS OF TERM STRUCTURE MODELS*

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RESUMO

Este artigo compara as habilidades de interpolação da estrutura a termo de modelos não-parametricos e paramétricos, utilizados pelos principais bancos centrais do mundo. Buscando a fusão de suavidade e flexibilidade, é introduzido um novo modelo de seis fatores à classe Nelson-Siegel. Ele surge como uma extensão natural dos de Svensson (1994) e de cinco fatores proposto por Rezende e Ferreira (2008) e Christensen, Diebold e Rudebusch (2008). Os resultados mostram a superioridade do modelo de smoothing spline sobre os demais, e vantagem do de seis fatores sobre os outros da classe Nelson-Siegel. Também é mostrado que a superioridade do modelo de smoothing spline, no entanto, vem com um custo: a sua instabilidade na interpolação dos primeiros vértices das curvas spot e forward. O de seis fatores, por outro lado, apresenta a propriedade desejável de suavidade e também uma grande flexibilidade, especialmente na interpolação das taxas forward e das taxas spot e forward de médio e longo prazos.

Palavras-Chave: Curvas Spot, Curvas Forward, modelos da classe Nelson-Siegel, smoothing spline

ABSTRACT

This paper compares the interpolation abilities of nonparametric and parametric term structure models which are widely used by the main Central Banks of the world. Seeking the fusion of smoothness and flexibility a new Nelson-Siegel class parametric model of six factors is introduced. It emerges as a natural extension of the Svensson (1994) and the five factor model proposed by Rezende and Ferreira (2008) and Christensen, Diebold and Rudebusch (2008). The results show the superiority of the smoothing spline model over the other ones in interpolating the spot and forward rates, and also the advantage of the proposed six factor model over the other ones of the Nelson-Siegel class. It is also shown that the superiority of the smoothing spline, however, comes with a cost: its instability in fitting the initial vertices of the term structure. The six factor model, on the other hand, exhibits the desirable property of smoothness and also an high flexibility, specially for the forward rates and for the spot and forward rates of medium and long terms.

Keywords: Spot Curve, Forward Curve, Nelson-Siegel class models, smoothing spline.

JEL Classification: E43, G12.

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* Comments and suggestions are welcome. Any remaining errors are my responsability only. This version: July 15, 2008.

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1 INTRODUCTION

In the last decades the use of the term structure of interest rates has been one of the most important topics of research in macroeconomics and finance. For macroeconomics, in a monetary policy context, forward rates are potentially useful as indicators of market expectations of future interest rates, inflation rates and exchange rates as discussed by Svensson (1994) and Sodelind and Svensson (1997) and the yield curve carries information about future GDP growth as shown by Estrella and Mishkin (1996, 1998). For finance, fixed income portfolio managers make use of the yield curve to mark to market, while risk managers use it for pricing derivatives and performing hedging operations. However, the market does not provide us securities at all the desired maturities and what we observe is only an incomplete set of yields across the maturity spectrum. This way, to overcome this problem, it is necessary some interpolation method. This exercise is what constitutes yield curve estimation.

Basically, the literature on term structure interpolation can be divided in the parametric and nonparametric methods. The parametrics, which the main representatives are the Nelson and Siegel (1987) and the Svensson (1994) models, exhibit at least three reasons for their popularity. First, they are easy to estimate. In fact, if the so-called time-decaying parameters are fixed, their curves are obtained by linear regression techniques. If not, one has to resort to non-linear regression methods. Second, adapting them in a time series context, it is possible to obtain accurate yield curve forecasts, and their estimated factors can assume economic interpretations of level, slope, curvature and double curvature of the yield curves [see Diebold and Li (2006), De Pooter (2007) and Almeida et al. (2007)]. Third, their functional forms impose more smoothness on the shapes of the curves, as desirable by macroeconomists [see Gürkaynak, Sack and Wright (2007)]. However, parametric methods are not immune to problems. First, they do not impose the presumably desirable theoretical restriction of absence of arbitrage [Filipovic (1999) and Diebold, Piazzesi and Rudebusch (2005)]. And second, they are not flexible enough to fit well both noise curves as curves with a long maturity spectrum.

The nonparametric methods, which the main representatives are the spline models developed by McCulloch (1971, 1975), Vasicek and Fong (1982) and Fisher, Nychka and Zervos (1995), also present some good properties. First, since they do not assume a particular functional form, they are robust to misspecification errors. Second, they exhibit great flexibility fitting almost perfectly all kinds of curves. The flexibility, however, comes with costs. The models constantly exhibit great instability on fitting, specially, in the extremes of the curves and the estimation involves a large nunber of parameters. Another problem is that the location and number of the knot points must be chosen.

This way, it can be concluded that when one must decide which interpolation method is going to be used, basically, one is confronted by the issue: how much flexibility to allow in the curve estimation. If a spline-based method is chosen, a very flexible curve could be estimated, but it would be done with considerable variability in the spot and forward rates. On the other hand, through the parametric methods, more smoothness could be imposed on the shapes of the curves, while some of the fit would be sacrified.

The choice in this dimension depends on the purpose that the curves are intended to serve. A trader looking for small pricing anomalies may be very concerned with how a specific security is priced relative to those securities immediately around it and would, probably, choose the more flexible method to estimate the yield curve. By contrast, a macroeconomist may be more interested in understanding the fundamental determinants of the yield curve and the expectations of some economic variables indicated by the forward curve, prefering then the smoothest method.

Trying to solve this puzzle this paper proposes a parametric method of six factors (hereafter SF) which is also flexible enough to fit accurately a pool of spot and forward curves shapes. Adding only two parameters in the estimation procedure, it´s shown that the proposed model, which can be included in the Nelson-Siegel class[1], presents a great flexibility gain, fitting very well all the yields through the maturity spectrum, but, specially, the longests. The results are compared with those obtained by the models of Fisher, Nychka and Zervos (1995) (hereafter SS), Nelson and Siegel (1987) (hereafter NS), Svensson (1994) (hereafter SV) and by the five factor parametric model (hereafter FF) proposed by Rezende and Ferreira (2008) and Christensen, Diebold and Rudebusch (2008)[2]. This choice was based on the conclusion that these models are widely used in Central Banks and industry[3] (the SS), including the Federal Reserve Board [see Gürkaynak, Sack and Wright (2007)], the European Central Bank [see Coroneo, Nyholm and Vidova-Koleva (2008)] and many other Central Banks [see Bank for International Settlements - BIS (2005)].

The remainder of the paper is organized as follows. The second section presents the models which will be analyzed in the paper; the third discusses the data used in the estimation; in the fourth section the estimation procedures of the models are addressed; the fifth section presents the results; and the sixth section concludes the paper.

2 TERM STRUCTURE MODELS

2.1 BASIC DEFINITIONS

The term structure of interest rates can be described in terms of the spot (or zero-coupon) rate, the discount rate and the forward rate. The forward curve determines rates as a function of maturities. A forward rate is the interest rate of a forward contract on an investment which will be initiated $\bar{\tau}$ periods in the future and which will mature τ^* periods beyond the start date of the contract. We obtain the instantaneous forward rate $f(\bar{\tau})$ by letting the maturity of such forward contract go to zero: $\lim_{\tau^* \to 0} f(\tau^*, \bar{\tau}) = f(\bar{\tau})$.

From the instantaneous forward rates, we get the forward curve, $f(\tau)$.

We can then determine the spot rate implicit in a zero-coupon bond with maturity $\bar{\tau}$, $y(\bar{\tau})$. Under continuous compounding, taking an average of forward rates, we get the spot rate:

$$y(\bar{\tau}) = \frac{1}{\tau}\int_0^{\tau} f(x)dx \qquad (2.1)$$

Then, from the spot rates, we get the spot yield curve, $y(\tau)$.

[1] This class includes the models of Nelson and Siegel (1987), Svensson (1994), Bliss (1997), Björk and Christensen (1999) and the five factor model presented by Christensen, Diebold and Rudebusch (2008).

[2] Christensen, Diebold and Rudebusch (2008) derived a model with no-arbitrage restrictions. In this paper, we consider the five factor model without the restrictions.

[3] The SS model

The discount curve is made by rates which gives the present value of a zero-coupon bond that pays a nominal value of $1.00 after τ periods. It can be obtained from the spot curve through the following relationship:

$$d(\tau) = e^{-y(\tau)\tau} \qquad (2.2)$$

From the equations above we can then relate the discount and the forward curves by the following formulas:

$$d(\tau) = \exp\left[-\int_0^{\tau} f(x)dx\right] \qquad (2.3)$$

$$f(\tau) = -\frac{d'(\tau)}{d(\tau)} \qquad (2.4)$$

We can move from a curve to the other using the relationships specified above.

2.2 NONPARAMETRIC MODEL

The smoothing spline was introduced by Fisher, Nychka and Zervos (1995). In general, for an explanatory variable x_i and a response variable y_i, this method tries to find a smooth function $f(.)$ to minimize the following functional:

$$Q_{\omega}(f) = \sum_{i=1}^{n}[y_i - f(x_i)]^2 + \omega\int[f''(t)]^2 dt \qquad (2.5)$$

It can be viewed as a penalized residual sum of squares, where the first term is the residual sum of squares (RSS) and the second term is the penalty term. In the last one, the parameter ω controls the trade-off between goodness-of-fit and parsimony. An increase in the penalty reduces the effective number of parameters to be estimated. Fisher, Nychka and Zervos (1995) suggested using generalized cross validation (GCV) to choose ω. That is, ω is chosen to minimize

$$GCV = \frac{RSS}{n - \theta.tr(S)} \qquad (2.6)$$

where θ is called the *cost*, *n* is the dimension of the implicit smoother matrix *S* and *tr(S)* denotes the trace of *S* and is usually used as the measure of the effective number of parameters. Hence, θ controls the entire parametrization of the spline. Following Fisher, Nychka and Zervos (1995) θ was preset in the value of 2.

2.3 PARAMETRIC MODELS

Nelson and Siegel (1987) suggest to fit the forward curve at a particular point in time using the following parametric model:

$$f(\tau) = \beta_1 + \beta_2 e^{-\tau/\lambda} + \beta_3 \,{}^{\tau}\!/_{\lambda}\, e^{-\tau/\lambda} \qquad (2.7)$$

From (2.1) we can get the spot yield curve:

$$y(\tau) = \beta_1 + \beta_2 \left(\frac{1 - e^{-\tau/\lambda}}{\tau/\lambda} \right) + \beta_3 \left(\frac{1 - e^{-\tau/\lambda}}{\tau/\lambda} - e^{-\tau/\lambda} \right) \qquad (2.8)$$

where the constant λ governs the decaying speed of the β_2´s exponential component and the maximum point of the β_3´s exponential component. Thus λ governs the decay rate of the whole curve. The exponential components of the spot and forward NS curves can be viewed in Figure 1 (a) and Figure 2 (a), respectively.

Although the basic model captures many curves shapes, it can not deal with all the shapes that the term structure assumes over time, specially the longer ones and those which use to appear twisted, with more than one inflection point. Trying to remedy this problem, several more flexible parametric models of the NS class have been proposed in the literature, adding additional factors, including other decaying parameters, or combining both of them.

A popular term structure approximation model is the four factor SV model. Svensson (1994) proposes to increase the NS flexibility through the inclusion of a fourth exponential component that recalls the third component of the basic one, presenting a different parameter λ. The model that fits the forward curve is given by:

$$f(\tau) = \beta_1 + \beta_2 e^{-\tau/\lambda_1} + \beta_3 \,{}^{\tau}\!/_{\lambda_1}\, e^{-\tau/\lambda_1} + \beta_4 \,{}^{\tau}\!/_{\lambda_2}\, e^{-\tau/\lambda_2} \qquad (2.9)$$

And the model that approximate the zero-coupon yield curves:

$$y(\tau) = \beta_1 + \beta_2 \left(\frac{1 - e^{-\tau/\lambda_1}}{\tau/\lambda_1} \right) + \beta_3 \left(\frac{1 - e^{-\tau/\lambda_1}}{\tau/\lambda_1} - e^{-\tau/\lambda_1} \right) + \beta_4 \left(\frac{1 - e^{-\tau/\lambda_2}}{\tau/\lambda_2} - e^{-\tau/\lambda_2} \right) \qquad (2.10)$$

The fourth component differs from the third only because of the decaying parameter λ. It can be interpreted as a double curvature component, as well its factor. The SV model, theoretically, fits the various spot and forward curves shapes better then the three factor model. The exponential components of the SV curves can be viewed in Figure 1 (b) and Figure 2 (b), respectively.

The five factor model of introduced by Rezende and Ferreira (2008) and by Christensen, Diebold and Rudebusch (2008) emerges as a natural extension of the SV. Seeking a greater flexibility they included another term, which recalls the second NS exponential component. It differs because of the decaying parameter. The following model fits the forward curve:

$$f(\tau)=\beta_1+\beta_2 e^{-\tau/\lambda_1}+\beta_3 e^{-\tau/\lambda_2}+\beta_4 \tau/\lambda_1\, e^{-\tau/\lambda_1}+\beta_5 \tau/\lambda_2\, e^{-\tau/\lambda_2} \tag{2.11}$$

And the one that models the spot curve is given by:

$$y(\tau)=\beta_1+\beta_2\left(\frac{1-e^{-\tau/\lambda_1}}{\tau/\lambda_1}\right)+\beta_3\left(\frac{1-e^{-\tau/\lambda_2}}{\tau/\lambda_2}\right)+\beta_4\left(\frac{1-e^{-\tau/\lambda_1}}{\tau/\lambda_1}-e^{-\tau/\lambda_1}\right)+\beta_5\left(\frac{1-e^{-\tau/\lambda_2}}{\tau/\lambda_2}-e^{-\tau/\lambda_2}\right) \tag{2.12}$$

The third component of both the curves can be interpreted as a double slope component and can be visualized in Figure 1 (c) and Figure 2 (c).

The proposed six factor model is also an extension of the other ones described above. Seeking a greater flexibility we included another term which is a modification of the third. t differs because of the decaying parameter. In its dynamic way, we proposed the following model to fit the forward curve:

$$f(\tau)=\beta_1+\beta_2 e^{-\tau/\lambda_1}+\beta_3 e^{-\tau/\lambda_2}+\beta_4 \tau/\lambda_1\, e^{-\tau/\lambda_1}+\beta_5 \tau/\lambda_2\, e^{-\tau/\lambda_2}+\beta_6\left[e^{-\tau/\lambda_1}+\left(2\tau/\lambda_1-1\right)e^{-2\tau/\lambda_1}\right] \tag{2.13}$$

And the one that models the spot curve:

$$y(\tau)=\beta_1+\beta_2\left(\frac{1-e^{-\tau/\lambda_1}}{\tau/\lambda_1}\right)+\beta_3\left(\frac{1-e^{-\tau/\lambda_2}}{\tau/\lambda_2}\right)+\beta_4\left(\frac{1-e^{-\tau/\lambda_1}}{\tau/\lambda_1}-e^{-\tau/\lambda_1}\right)+\beta_5\left(\frac{1-e^{-\tau/\lambda_2}}{\tau/\lambda_2}-e^{-\tau/\lambda_2}\right)+\beta_6\left(\frac{1-e^{-\tau/\lambda_1}}{\tau/\lambda_1}-e^{-2\tau/\lambda_1}\right)$$

(2.14)

The sixth component can be interpreted as a triple curvature component. However it presents a bigger maximum point. The exponential components of both the SF curves can be visualized in Figure 1 (d) and Figure 2 (d). We expect that the six factor model fits better more complex and twisted forward and yield curves, like those with two or more inflection points. We also expect that the greater flexibility allows for a better fit at the short and long term maturities of the term structure.

3 DATA

The data set used in in the estimations are the monthly spot interest rates and the corresponding instantaneous forward rates of the McCulloch U.S. Treasury term structure data[4]. All rates are end-of-month, given as percentages per annum, and are on a continuous-compounding basis. They are derived from a tax-adjusted cubic spline discount function, as described in McCulloch (1975). We considered a data with 73 curves which present 48 maturities given in years: 0.083, 0.167, 0.25, 0.333, 0.417, 0.5, 0.583, 0.667, 0.75, 0.833,

[4] The data can be downloaded from http://www.econ.ohio-state.edu/jhm/ts/mcckwon/mccull.htm.

0.917, 1, 1.083, 1.167, 1.250, 1.333, 1.417, 1.5, 1.75, 2, 2.5, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28. The sample goes from February 1985 to February 1981. The Figure 3 shows the forward and spot curves along the sample.

The McCulloch data was chosen because one of the interests of the paper is to verify the fitting of the models to spot and forward curves with a long-end. This permits us to verify how flexible they are. As known, this is one of the main difficulties of the NS class models.

4 ESTIMATION PROCEDURE

The parameters λ that govern the behavior of the exponential components of each parametric model were fixed to facilitate the estimations[5]. For NS, in a vector of possible optimal parameters λ, one was chosen. It provided the lowest average term structure fitting error, measured by the average of the Root Mean Squared Error - RMSE. Explaining in a better way, initially a vector of parameters λ was created and, for each element of that vector, the factor loadings were fixed. Then, for each λ, a daily cross-sectional OLS was applied to the model, obtaining its factors time series as in Diebold and Li (2006). Multiplying the estimated factors by the pre-fixed loadings we then get the fitted term structure by the NS model for each λ. The RMSE was then calculated for each term structure maturity and its averages was taken. Doing this we have obtained an average term structure RMSE for each element of the vector of parameters λ, choosing, finally, the one (the optimal parameter) that generates the lowest RMSE.

The same criteria for the selection of the decaying parameters was adopted to the SV, FF and SF models. The difference was that two different parameters determine their factor loadings. Thus many possible combinations between λ_1 e λ_2 were created, choosing the one that generated the lowest average RMSE to the fitting of the whole term structure. The estimation process was also the same. The optimal parameters, for each curve and model, are shown in Table 1.

5 RESULTS

The Table 2 provides the average, and by maturities, spot curve fitting errors of each model, measured by the RMSE criterion. As expected, due the greater flexibility, the SS model presents a large advantage over the other ones. Interesting to note, however, is the superiority of the SF over the other NS class models. For the majority of the maturities the errors are substantially smaller, showing the flexibility gain obtained with the inclusion of the third curvature term. This advantage occurs in the maturity spectrum and also in the time spectrum, as shown by the Figures 4 (a) and (c). In the maturity spectrum, the SF is better than the NS and SV models, specially, in the medium and long term vertices. In relation to the FF, its superiority is higher until the vertice 15. In the time spectrum, the SF is better than the NS and SV along all the sample and, in general, is also better than the FF.

However, the advantage of the SF over the other parametric models is more notable for the forward curves, as shown by the Table 3 and Figures 4 (b) and (d). In the time

[5] Despite this procedure reduce the flexibility of the parametric models, it does not compromise the comparison between them.

domain the superiority along all the sample is apparent, and in the maturity domain it is more apparent in the vertices that goes from 7 to 28. The fitting differences are greater in the medium and long term maturities and they clearly influence the outcome of the average RMSE. Notice that the SF is 370% more accurate, in average, than the FF, an extraordinary result. The results also show that the difference between the SF and SS models is smaller in the forward curves modeling.

From the results described above the importance of the inclusion of the third curvature component seems to be clear. To atest it, along all the sample, the adjusted R^2 statistics were calculated for the cross-section regressions of the NS class models. The Figures 4 (e) and (f) show the results. We observe that the new component adds information for the spot and forward curves modeling. The results are notable for the forward rates. In general, the NS, SV and FF exhibit a poor fit, but the SF does not. Its R^2 statistics are superior than 80% in the entire sample, and are superior than 95% in 58 of the 73 forward curves of the sample.

The Figures 5 and 6 show some yield and forward curves examples in specific months of the sample, fitted by all the analysed models. As pointed above, the SS model interpolates the curves better. Note also the advantage of the SF over the other NS class models, specially for the forward curves and for the medium and long term vertices of both of them. The superiority of the SS, however, comes with a cost: its instability in fitting some parts of the term structure. The Figures 5 (c), (d) and (f) and the Figures 6 (c), (d), (e) and (f) clearly show its weakness. The SS is very unstable in interpolating the beginning of both curves. On the other hand, the SF seems to interporlate the spot and forward rates with an high smoothness and also with a good flexibility.

6 CONCLUSIONS

This paper compares the interpolation abilities of the most widely nonparametric and parametric term structure models used by the main Central Banks of the world. Seeking the fusion of smoothness and flexibility a new NS class parametric model is introduced. It emerges as a natural extension of the SV and FF models, proposed by Svensson (1994) and Rezende and Ferreira (2008) and Christensen, Diebold and Rudebusch (2008).

The results show the superiority of the SS model over the other ones in interpolating the spot and forward rates, and also the advantage of the proposed SF model over the other ones of the NS class. It is also shown that the superiority of the SS, however, comes with a cost: its instability in fitting the initial vertices of the term structure. The SF, on the other hand, exhibit the desirable property of smoothness and also an high flexibility, specially for the forward curves and for the medium and long term maturities of both the curves.

Despite the smoothness is important for macroeconomics purposes, the flexibility is also a desirable property. The poor construction of the yield and forward curves can imply in the wrong understanding and measurement of important economic informations carried by the term structure, specially those used for monetary policy purposes. Hence, the insertion of flexibility in a class of models largely used by many Central Banks around the world, like the NS class, can improve the conduction of the monetary policy. This flexibility gain can also make the NS models most usable in industry.

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Table 1: Optimal parameters λ

Models	Curves	
	Spot	Forward
NS	1.882	0.304
SV	0.714; 3.345	1.983; 0.673
FF	4.584; 1879.5	1.871; 0.277
SF	2.086; 1919.89	24.356; 15.167

Notes: This table shows the optimal decaying parameters obtained in the estimation procedure of the NS class models.

Table 2: Spot Curve Fitting - RMSE

Maturities in years	Models				
	SS	NS	SV	FF	SF
0.083	0.00018	0.00205	0.00107	0.00222	0.00100
0.25	0.00017	0.00082	0.00038	0.00088	0.00035
0.5	0.00008	0.00051	0.00052	0.00046	0.00045
0.75	0.00009	0.00074	0.00053	0.00075	0.00042
1	0.00008	0.00075	0.00032	0.00074	0.00032
1.5	0.00024	0.00061	0.00047	0.00065	0.00045
2	0.00039	0.00073	0.00071	0.00069	0.00062
3	0.00019	0.00087	0.00074	0.00086	0.00045
4	0.00026	0.00091	0.00077	0.00102	0.00042
5	0.00009	0.00082	0.00066	0.00093	0.00041
7	0.00005	0.00071	0.00066	0.00063	0.00046
10	0.00003	0.00088	0.00068	0.00056	0.00046
12	0.00003	0.00099	0.00070	0.00044	0.00020
15	0.00002	0.00111	0.00103	0.00065	0.00066
17	0.00001	0.00105	0.00110	0.00060	0.00061
20	0.00001	0.00122	0.00122	0.00026	0.00017
23	0.00001	0.00152	0.00123	0.00071	0.00076
26	0.00002	0.00143	0.00114	0.00032	0.00031
28	0.00005	0.00294	0.00309	0.00111	0.00125
Mean	0.00009	0.00100	0.00080	0.00067	0.00047

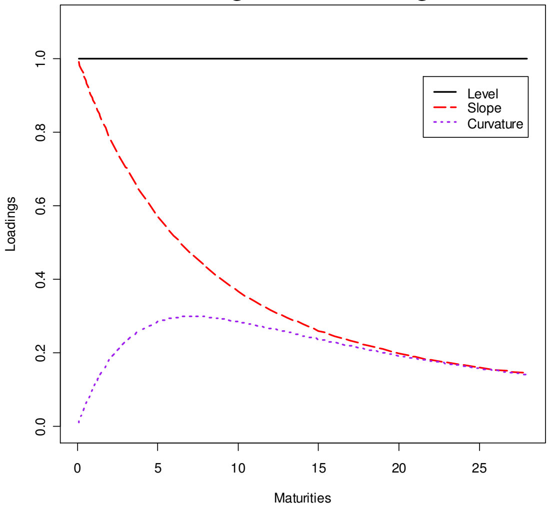
Notes: This table shows the fitting RMSE, average and by maturities, of the SS, NS, SV, FF and SF models to the spot curves.

Table 3: Forward Curve Fitting - RMSE

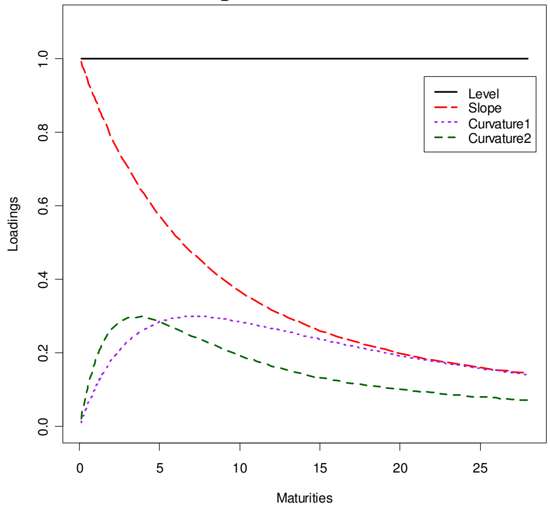
Maturities in years	Models				
	SS	NS	SV	FF	SF
0.083	0.00238	0.00211	0.00154	0.00109	0.00360
0.25	0.00126	0.00146	0.00158	0.00158	0.00154
0.5	0.00111	0.00262	0.00189	0.00152	0.00192
0.75	0.00103	0.00191	0.00125	0.00130	0.00174
1	0.00097	0.00172	0.00148	0.00164	0.00187
1.5	0.00145	0.00215	0.00155	0.00175	0.00170
2	0.00143	0.00264	0.00174	0.00294	0.00177
3	0.00242	0.00394	0.00266	0.00231	0.00282
4	0.00162	0.00377	0.00220	0.00210	0.00267
5	0.00143	0.00378	0.00227	0.00293	0.00270
7	0.00125	0.00357	0.00313	0.00468	0.00211
10	0.00027	0.00462	0.00470	0.00662	0.00091
12	0.00037	0.00625	0.00659	0.00889	0.00142
15	0.00022	0.00837	0.00885	0.00889	0.00091
17	0.00027	0.01033	0.01074	0.01078	0.00109
20	0.00025	0.01193	0.01211	0.01213	0.00135
23	0.00033	0.00824	0.00786	0.00784	0.00080
26	0.00053	0.02134	0.02088	0.02084	0.00203
28	0.00081	0.04295	0.04268	0.04265	0.00255
Mean	0.00091	0.00657	0.00627	0.00620	0.00166

Notes: This table shows the fitting RMSE, average and by maturities, of the SS, NS, SV, FF and SF models to the forward curves.

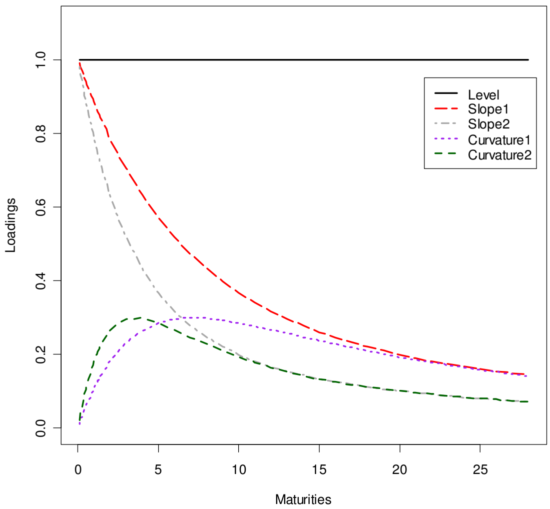
Figure 1: Loadings of the NS Class Models – Spot Curve



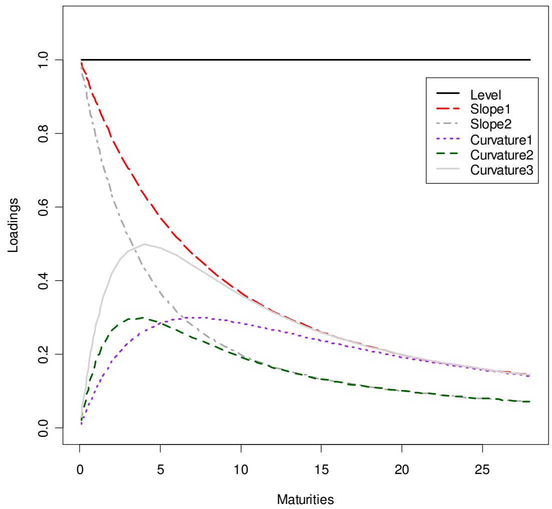
(a) NS Model



(b) SV Model



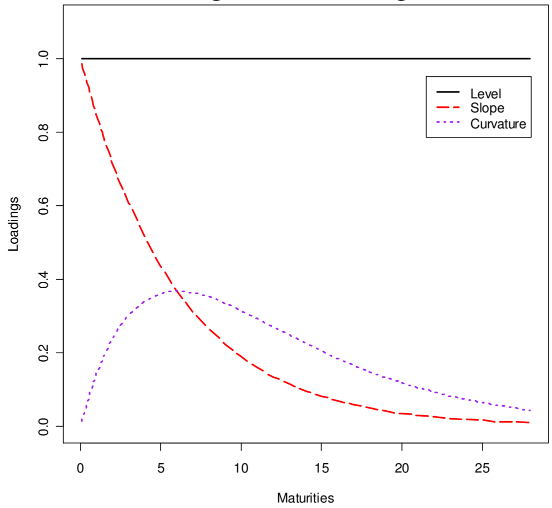
(c) FF Model



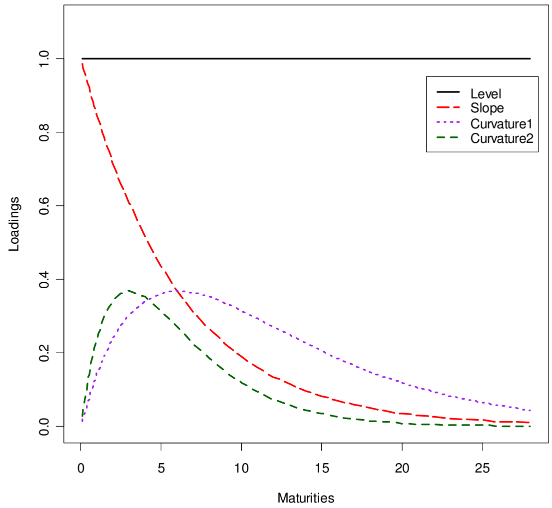
(d) SF Model

Notes: This Figure exhibit the loadings of the NS models of the yield curve.

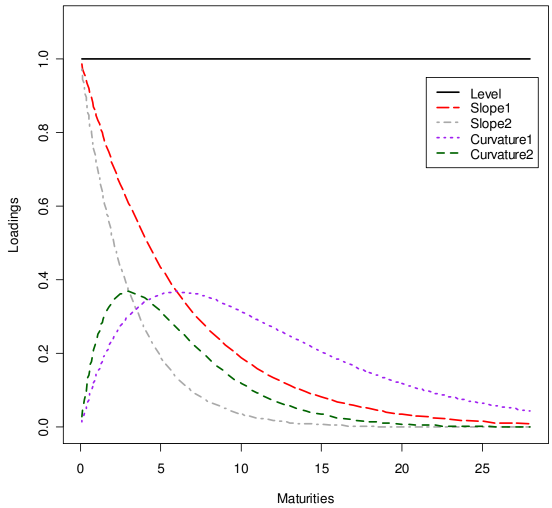
Figure 2: Loadings of the NS Class Models – Forward Curve



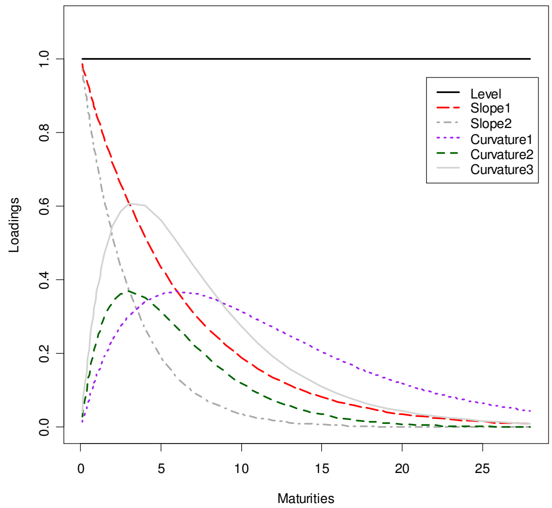
(a) NS Model



(b) SV Model



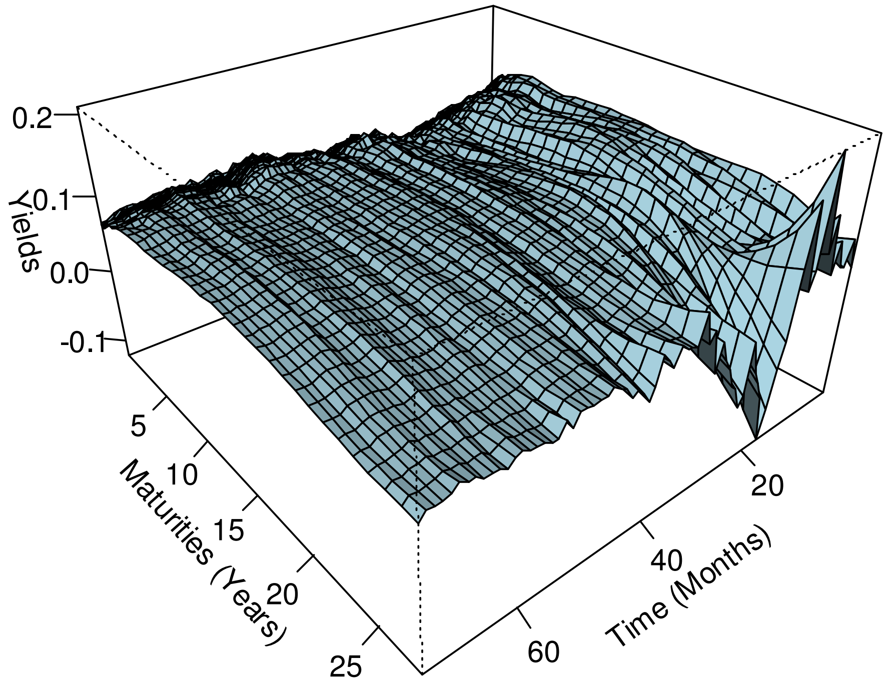
(c) FF Model



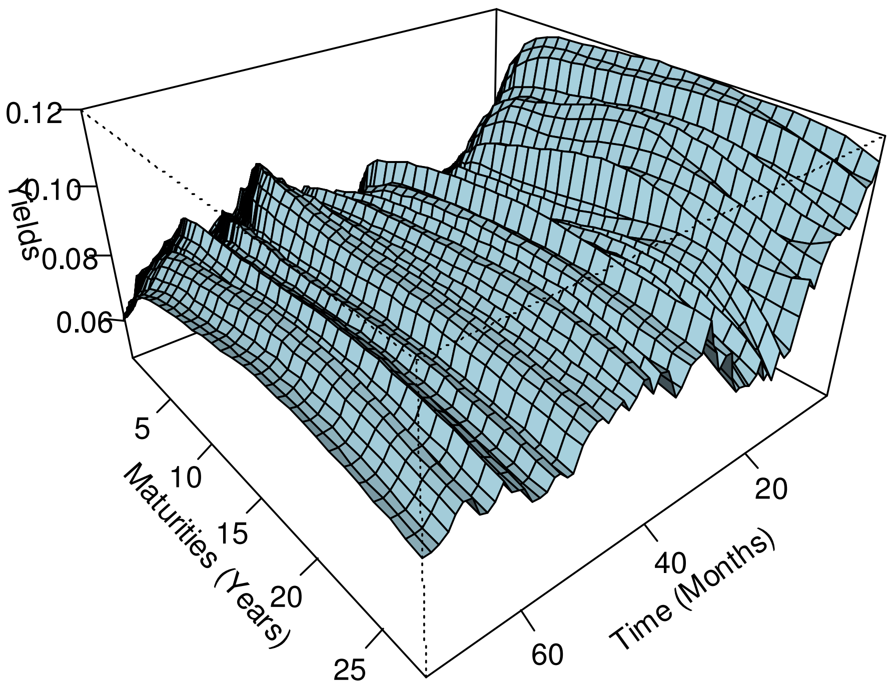
(d) SF Model

Notes: This Figure exhibit the loadings of the NS models of the forward curve.

Figure 3: Spot and Forward Curves



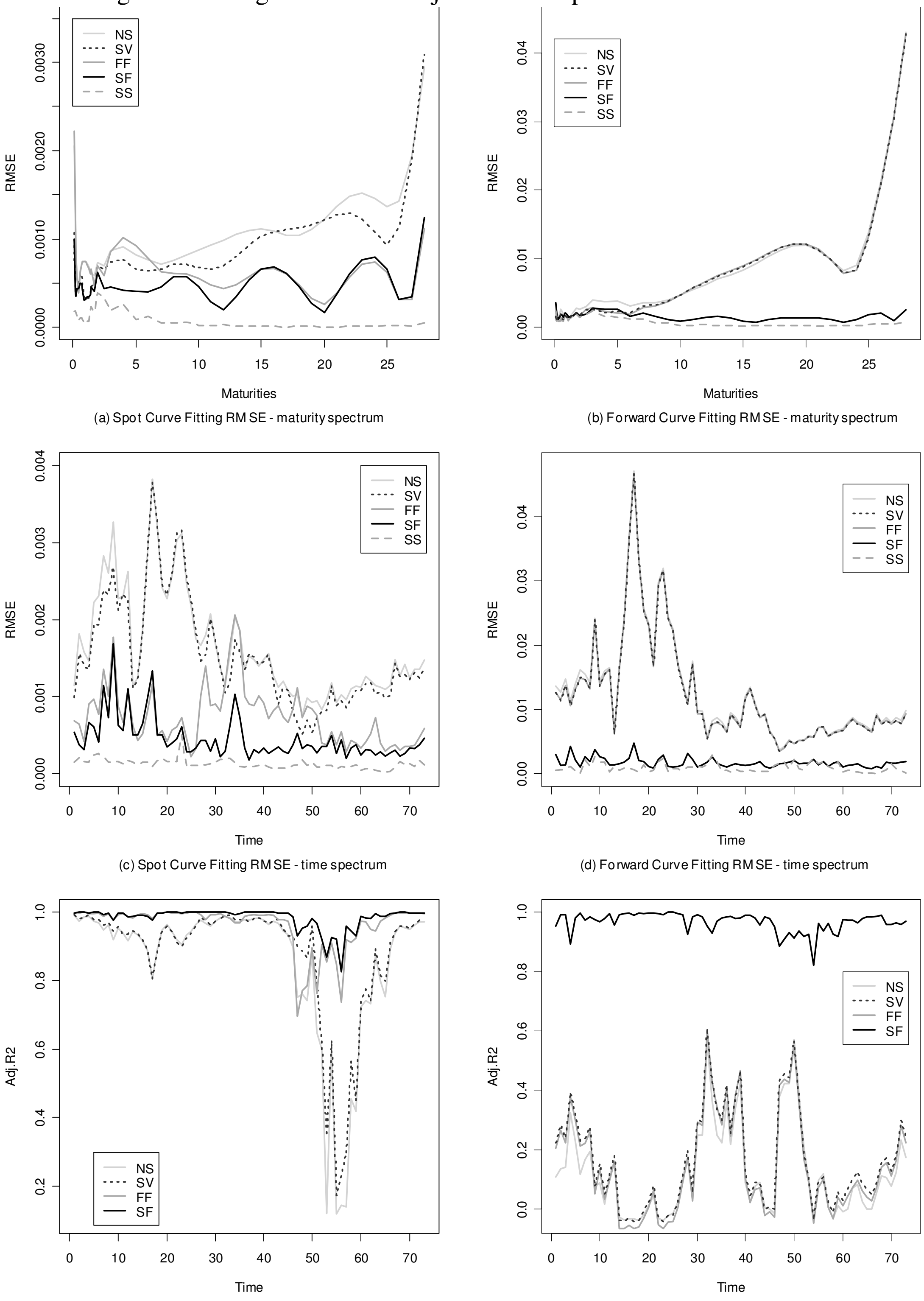
(a) Forward Curves



(b) Spot Curves

*Notes:*The Figure 3 (a) and the Figure 3 (b) shows the US Treasury Forward and Spot Curves, respectvely, of the McCulloch data. The sample goes from February 1985 to February 1981.

Figure 4: Fitting RMSE and Adjusted R^2 – Spot and Forward Curves



(a) Spot Curve Fitting RMSE - maturity spectrum

(b) Forward Curve Fitting RMSE - maturity spectrum

(c) Spot Curve Fitting RMSE - time spectrum

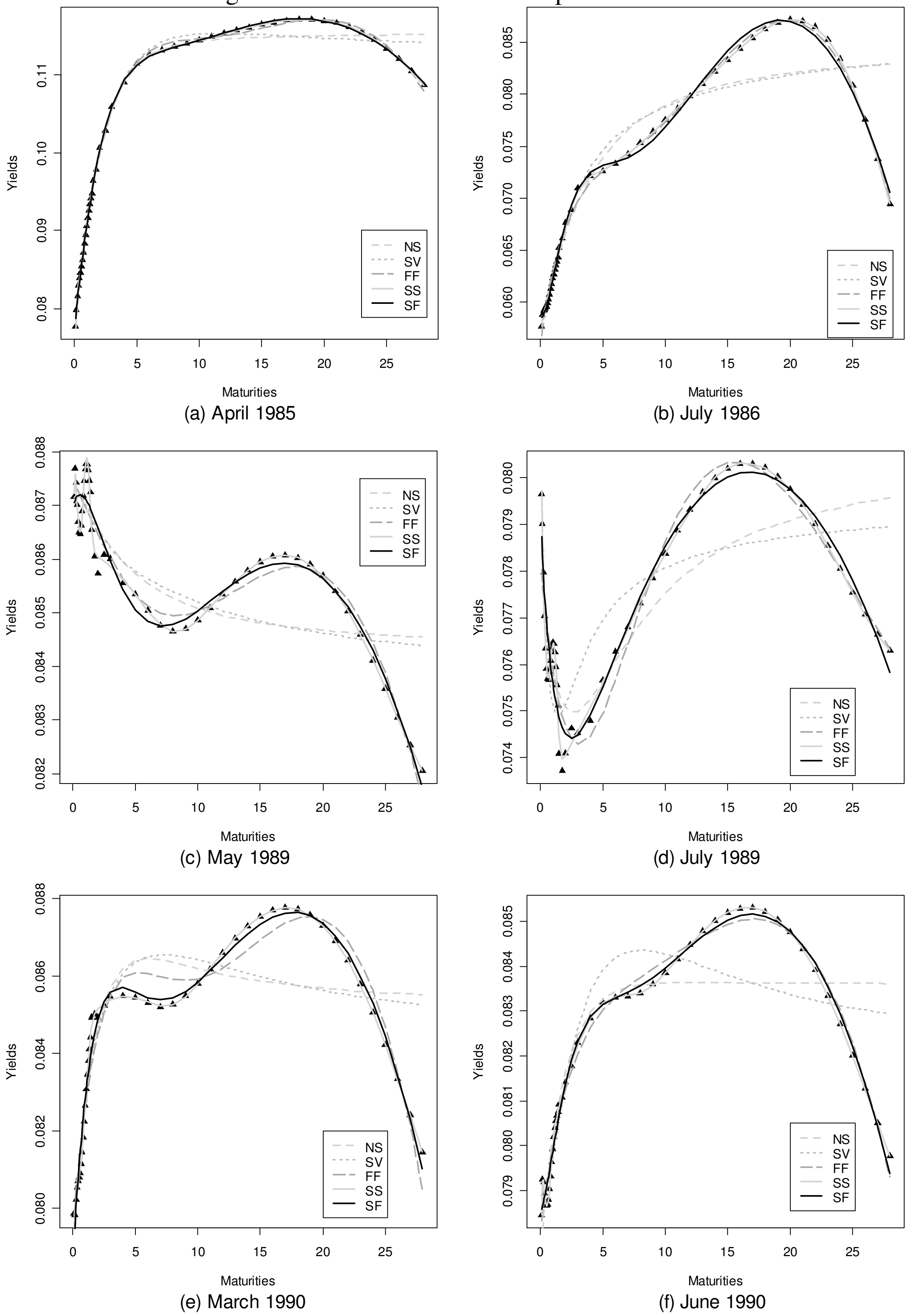
(d) Forward Curve Fitting RMSE - time spectrum

(d) Adjusted R2 - Spot Curve

(e) Adjusted R2 - Forward Curve

*Notes:*This Figure shows the spot and forward curves fitting RMSE in the maturity and time spectrum. It also shows the Adjusted R2 statistics of the cross-section regression of the NS class models for all the curves of the sample.

Figure 5: Fitted Yield Curves in specific months



(a) April 1985

(b) July 1986

(c) May 1989

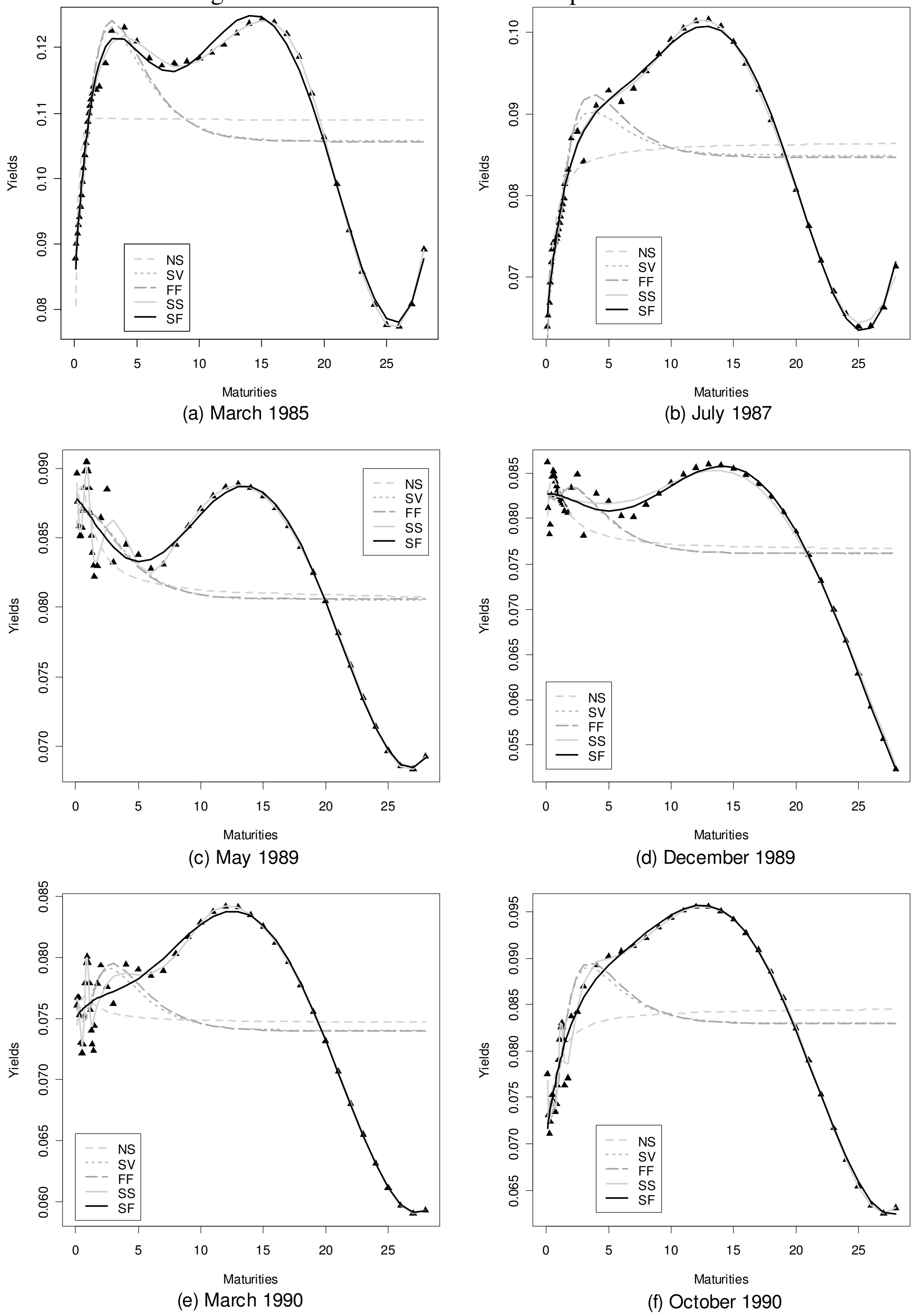
(d) July 1989

(e) March 1990

(f) June 1990

Notes: The Figure shows the McCulloch spot curves observed in six specific months of the sample and exhibit the fitting of the SS, NS, SV, FF and SF models to the observed curves.

Figure 6: Fitted Forward Curves in specific months



Notes: The Figure shows the McCulloch forward curves observed in six specific months of the sample and exhibit the fitting of the SS, NS, SV, FF and SF models to the observed curves.