ARROW-DEBREU AND THE LAW OF DIMINISHING RETURNS:
A CRITICAL APPRAISAL

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RESUMO

Este artigo discute a idéia de que a lei dos retornos marginais decrescentes pode ser derivada da teoria de equilíbrio geral de Arrow-Debreu se a noção de livre concorrência é aceita. Demonstra-se que, se os conjuntos das possibilidades de produção de todas as mercadorias forem dados, então o conceito de equilíbrio de longo prazo é suficiente para se determinar todos os preços de equilíbrio, independentemente do processo de maximização de preços, o que é desnecessário para este propósito. Além disso, prova-se que, pressupondo-se livre concorrência, a lei dos rendimentos marginais decrescentes não pode ser derivada, de forma que os fenômenos de reversão de técnicas e aprofundamento inverso do capital não podem ser excluídos mesmo em se aceitando as hipóteses tradicionais do modelo de Arrow-Debreu a respeito dos conjuntos das possibilidades de produção.

Palavras-chave: equilíbrio geral; modelo de Arrow-Debreu; rendimentos decrescentes; concorrência; reversão de técnicas; aprofundamento inverso do capital.

ABSTRACT

This article challenges the notion that the law of diminishing returns can be derived from the standard Arrow-Debreu theory of general equilibrium if the notion of free competition is accepted. It shows that, if the production sets of all commodities are given, the traditional concept of long-run equilibrium is sufficient to have all long-run equilibrium prices determined regardless of profit maximization, which is not necessary for such purposes. Besides, it proves that, if competition is assumed, the law of diminishing marginal returns can not be derived so that reswitching of techniques and inverse capital deepening are indeed phenomena that can not be excluded even if the standard assumptions of the Arrow-Debreu model regarding the production sets are accepted.

Key-words: general equilibrium; Arrow-Debreu model; decreasing returns; competition; reswitching of techniques; inverse capital deepening

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1. Introduction and Objectives

In spite of Samuelson's (1966, p. 568) warning that "the simple tale told by Jevons, Böhm-Bawerk, Wicksell, and other neoclassical writers – alleging that, as the interest rate falls in consequence of abstention of present consumption in favour of future, technology must become in some sense more 'roundabout', more 'mechanised' and more 'productive' – cannot be universally valid", the very idea that the marginal productivity of capital is a monotonic decreasing function of the its intensity continues to be used by mainstream economists as an decisive tool for economic analysis. So the concept of aggregate production function, regardless of the available proof that "in a world where production is carried out using heterogeneous commodities it is impossible to define an aggregate measure of capital which, taken together with the other 'factors' of production, allows the determination of the level and distribution of the social output" (Baldone, 1984, p. 271).[1] A good example is the modern neoclassical theory of economic development and long run growth. Inspired in the seminal works of Lucas (1988) and Romer (1986, 1990) and available in a many books (see, for example, Barro and Sala-i-Martin, 1995; Romer, 1996; Jones, 1998), the neoclassical theory of economic development ignores such criticisms by assuming not only an aggregate production function with constant returns to scale, but also the law of decreasing marginal returns.

Initially, such persistence in using apparently inconsistent concepts took roots in the belief that the problems with the neoclassical theory of value and distribution – reswitching of techniques and reverse capital deepening – could be overcome by either an adequate comprehension of them or a correct specification of the hypothesis regarding the technology of production. However, this was not the case and, while Bruno, Burmeister, and Schleshinski's (1966) parallel between reswitching of techniques and multiple equilibria was contested by Kurz (1985), on the grounds that the phenomenon of multiple internal rates of return is a question within the partial framework of microeconomic theory of investment, and reswitching presupposes a total general framework, Sato (1974a and 1974b) and Hatta's (1976 and 1990) technical requirements were discharged because there can not be found any reason for "regular" economies" to exclude technologies that do not present "well behaved" results (Kurz and Salvadori, 1995, pp. 450-451, and Zambelli, 2004).[2] Finally, the contention formulated by Starret (1969), Burmeister and Dobell (1970), Stiglitz (1973), and Bliss (1975) that smooth production functions governing the columns of the input-output matrices were sufficient to prevent both reswitching of techniques and reverse capital deepening were disproved by Belino (1993), while Marglin's (1984) proof regarding the impossibility of reswitching was shown to be mistaken by Gontijo (1998).

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[1] In Hahn's (1982, p. 373) words, "there is no valid aggregation of wheat and barley into something called capital".

[2] Burmeister and Turnovsky (1972) define as "regular" an economy that exhibits "capital deepening response" for every admissible profit rate, i.e., for which a fall in the rate of profit results in an increase in the steady-state capital-labor ratio. Burmeister (1980

Notwithstanding the failure of all attempts to prove that both reswitching of techniques and reverse capital deepening are not quite "normal" phenomena, one possible reason for the reluctance of mainstream economists to abandon the "law of diminishing returns" and the concept of production function seems to be the belief that the modern general equilibrium theory developed by Arrow, Debreu, McKenzie and others furnishes a consistent foundation for the neoclassical approach, which, as a consequence, is immune to any major criticism, like those inspired by Joan Robinson (1954) and Sraffa (1960).[3] After all, according to main stream economists, Sraffa's arguments are irrelevant any way (Bliss, 1975; Hahn, 1975 and 1982; Burmeister, 1980), and the Sraffian theory can be viewed as a particular case of a more general, Arrow-Debreu type of general-equilibrium model (Nuti, 1976; Hahn, 1982).

The objective of this article is to exam this assumption, particularly with regard to the working of the law of diminishing marginal returns – it aims to show whether or not an inverse relationship between the quantity of a productive factor and its rate of rewards can be derived from the standard Arrow-Debreu model. Although it does not address all claims made by Hahn (1982) in his attempt to save neoclassical economics, its results can be used to reinforce the arguments raised by Duménil and Lévy (1985), Garegnani (1976, 1990), Kurz (1985) and Kurz and Salvadori (1995: 427-67) against crucial neoclassical tenets.

For the sake of simplicity, McKenzie's version of the general equilibrium theory is not discussed.

The discussion below is divided into five parts. Section 2 presents the assumptions of the Arrow-Debreu model of general equilibrium with respect to production and entrepreneurial behavior. Section 3 discusses the determination of the rate of return on capital. The law of diminishing marginal returns is examined in section 4. Section 5 presents the conclusions.

2. **Production Sets and Profit Maximization**

The standard Arrow-Debreu theory of production and profit maximization can be described as follows: taken an economy with n commodities, m producers and q primary factors of production, the commodity and primary factors space is thus an (n+q)-dimensional Euclidean space, denoted by R^{n+q}. Each producer **f** that produces a commodity **j** is endowed with a technology, denoted by $\mathbf{Y}_{fj}$, which lies in R^{n+q} and which constitutes the set of feasible plans. A production plan is a point $\mathbf{y}_{fj}$ that is feasible, which is expressed as $\mathbf{y}_{fj} \in \mathbf{Y}_{fj}$. A production plan is a specification of all inputs, primary factors, and outputs that are related by a given technology; outputs are represented by positive numbers, inputs and primary factors by negative numbers. For the sake of simplicity, joint production is assumed away, so that each $\mathbf{y}_{fj}$ has only one positive entry and all others are non-positive.

For the production set of producer **f**, $\mathbf{Y}_{fj}$, it is further assumed that

(1) $\mathbf{0} \in \mathbf{Y}_{fj}$;

(2) $\mathbf{Y}_{fj} - \{\mathbf{0}\} \neq \varnothing$;

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[3] According to Mainwaring (1984: 89), "there is, in Sraffa, no logical basis for a critique of sophisticated neoclassical theory based on the work of Arrow, Debreu and McKenzie."

(3) $\mathbf{Y}_{fj}$ is closed;

(4) $\mathbf{Y}_{fj}$ is bounded;

(5) $(-\Omega) \cap \mathbf{Y}_{fj} \neq \varnothing$;, where Ω is the non-positive cone of R^n;

(6) $\mathbf{Y}_{fj} \cap \Omega \subset \{\mathbf{0}\}$;

(7) $\mathbf{Y}_{fj} \cap (-\mathbf{Y}_{fj}) \subset \{\mathbf{0}\}$;

(8) $\mathbf{Y}_{fj} \cap \mathbf{Y}_{fk} \subset (-\Omega)$, where $j \neq k$.

(9) If $\mathbf{y}_{fj}$ is a boundary point of $\mathbf{Y}_{fj}$ in $(\pm \Omega_j)$, then $\lambda\mathbf{y}_{fj} \notin \mathbf{Y}_{fj}$ for all $\lambda > 1$, where $(\pm \Omega_j)$ is the semi-positive orthant of R^n with non-empty interception with $\mathbf{Y}_{fj}$.

Assumption (1) incorporates the possibility of inaction to the model, and assumption (2) ensures that the firm has always something planned to do. Closeness means that a production plan is feasible if a sequence of feasible production plans converges toward it and, so, it includes its boundary. Boundedness refers to the notion that resources are limited, which constrain production possibilities of each producer. Assumptions (5), (6), and (7) mean, respectively, free disposal, no "free lunch" (no commodity can be produced without use of inputs), and irreversibility of the production process. Assumption (8) prohibits joint production, and assumption (9) implies that the production technology $\mathbf{Y}_{fj}$ exhibits decreasing returns to scale, which means not only that $\mathbf{Y}_{fj}$ is convex, but that the strictly convex hull of $\mathbf{Y}_{fj} \cap (\pm \Omega_j)$, i.e. the smallest strictly convex set that contains $\mathbf{Y}_{fj} \cap (\pm \Omega_j)$, denoted by $C(\mathbf{Y}_{fj} \cap (\pm \Omega_j))$, is itself contained in the convex set $\mathbf{Y}_{fj} \cap (\pm \Omega_j)$.

For the total production set of commodity $\mathbf{j}$, or activity set $\mathbf{Y}_j = \Sigma_f \mathbf{Y}_{fj}$, which describes the production possibilities of commodity $\mathbf{j}$ of the whole economy, it is assumed the possibility of inaction; free disposal; no free lunch, and boundedness:

(10) $\mathbf{0} \in \mathbf{Y}_j$;

(11) $(-\Omega) \subset \mathbf{Y}_j$;

(12) $\mathbf{Y}_j \cap \Omega \subset \{\mathbf{0}\}$;

(13) $\mathbf{e}_j + \mathbf{Y}_j \geq \mathbf{0}$;

where $\mathbf{e}_j \geq \mathbf{0}$ is the vector of (limited) initial endowments.

Since the commodity space has finite dimension, assumptions (3) and (4) ensures that the producer set, $\mathbf{Y}_{fj}$, is compact. Because $\mathbf{Y}_{fj}$ is compact and the sum of n compact convex sets in R^n is compact and convex, and considering that compactness may be decomposed into closeness and boundedness, we can say that:

(14) $\mathbf{Y}_j$ is closed;

(15) $\mathbf{Y}_j$ is bounded;

(16) if $\mathbf{y}_j$ is a boundary point of $\mathbf{Y}_j$ in $(\pm \Omega_j)$, then $\lambda\mathbf{y}_j \notin \mathbf{Y}_j$ for all $\lambda > 1$, which implies that $C(\mathbf{Y}_j \cap (\pm\Omega_j)) \subset \mathbf{Y}_j \cap (\pm \Omega_j)$, i.e., $\mathbf{Y}_j$ exhibits decreasing returns to scale.

Finally, closeness, convexity and free disposal imply that feasible total production is one in which no output is larger and no output is smaller (in absolute value):

(17) $(\mathbf{Y}_j \cap - \Omega) \subset \mathbf{Y}_j.$

It is assumed that the producer $\mathbf{f}$ chooses from the set of feasible plans, $\mathbf{Y}_{fj}$; those plans that maximize his profit regarding commodity $\mathbf{j}$, defined as $\mathbf{p}\ \mathbf{y}_{fj}$, where $\mathbf{p}$ is the row vector of prices. The hypothesis of perfect competition ensures that $\mathbf{p}$ is taken as given. Producer $\mathbf{f}$ must then face the problem of choosing $\mathbf{y}_{fj}$ from $\mathbf{Y}_{fj}$ as to maximize $\mathbf{py}_{fj}$, subject to a feasibility constraint given by $\mathbf{Y}_{fj}$. Thanks to the Weierstrass theorem, which states that in finite-dimensional spaces a continuous function defined on a closed and bounded (compact) set has a maximum, this problem has a solution – an equilibrium production of the producer relative to $\mathbf{p}$. Note that if $\mathbf{y}_{fj}^*$ is a maximizer and $\mathbf{p} \neq \mathbf{0}$, the production set $\mathbf{Y}_{fj}$ is contained in the closed half-space below the closed supporting hyper-plane $\mathbf{H}$ that is tangent to it at $\mathbf{y}_{fj}^*$, with normal $\mathbf{p}$.

The profit function $\pi_{fj}(\mathbf{p})$ of firm $\mathbf{f}$ in relation to commodity $\mathbf{j}$ is defined as follows:

(18) $\pi_{fj}(\mathbf{p}) = \text{Max } \mathbf{p}\ \mathbf{y}_{fj}, \mathbf{y}_{fj} \in \mathbf{Y}_{fj}.$

Because $\mathbf{Y}_{fj}$ is closed and bounded and has full dimensionality (thanks to free disposal), $\pi_{fj}(\mathbf{p})$ is a continuous strictly convex function over the set of prices.

Since $C(\mathbf{Y}_{fj} \cap (\pm \Omega_j))$ is strictly convex and it is contained in $\mathbf{Y}_{fj}$ in the semi-positive orthant $(\pm \Omega_j)$, it can be defined the supply function of producer $\mathbf{j}$ regarding to commodity $\mathbf{j}$, $\mathbf{Y}_{fj}(\mathbf{p})$, as follows:

(19) $\mathbf{Y}_{fj}(\mathbf{p}) = \{\mathbf{y}_{fj}; \mathbf{p}\ \mathbf{y}_{fj} = \pi_{fj}(\mathbf{p}), \mathbf{y}_{fj} \in \mathbf{Y}_{fj}\}$

which is also continuous.

Considering that the total supply function of commodity $\mathbf{j}$ is defined as $\Sigma_f \mathbf{Y}_{fj}(\mathbf{p}) = \mathbf{Y}_j(\mathbf{p})$ and the total profit function as $\Sigma_{fj} \pi_{fj}(\mathbf{p}) = \pi_j(\mathbf{p})$, for a given $\mathbf{p}$, $\mathbf{y}_j = \Sigma_f \mathbf{y}_{fj}$ maximizes total profits on $\mathbf{Y}_j = \Sigma_f \mathbf{Y}_{fj}$ if and only if each $\mathbf{y}_{fj}$ maximizes profit on each $\mathbf{Y}_{fj}$. Both total profit and total supply functions are continuous, and the total profits function is strictly convex.

Strict convexity also ensures that there is a relevant range of $\mathbf{p}$, where the price vector is normal to the activity set $\mathbf{Y}_j$, which is contained in the closed half-space below the hyperplane that is tangent to it at the equilibrium production point $\mathbf{y}_j^*$.[4] The amplitude of this range depends on the availability of resources, which is given by $\mathbf{e}$. It is not difficult to see that in this range the total supply function is also a one-to-one function of $\mathbf{p}$ on $\mathbf{Y}_f$.

Thanks to Hotelling's lemma, if $\mathbf{Y}_j(\mathbf{p}_0)$ consists of a single point, then $\pi_j(\mathbf{p})$ is differentiable at $\mathbf{Y}_j(\mathbf{p}_0)$, which ensures that $D\mathbf{Y}_j(\mathbf{p}_{j0}) = D^2\mathbf{Y}_j(\mathbf{p}_{j0})$ is a symmetric and positive definite matrix with $D\mathbf{Y}_j(\mathbf{p}_{j0})\ \mathbf{p}_{j0} = \mathbf{0}$, which means that we have:

(20) $\partial\mathbf{Y}_j(\mathbf{p})\partial p_j > 0$ for all $j \in \mathbf{Y}_j$;

Note that $\mathbf{p} \neq \infty$ and $\mathbf{p} \neq \mathbf{0}$.

(21) $\partial \mathbf{Y}_j(\mathbf{p})\partial p_k = \partial \mathbf{Y}_k(\mathbf{p})\partial p_j < 0$ for all j, k $\in \mathbf{Y}_j$, j $\pm$ k.

Condition (21) is important, because it shows that quantities respond in the same direction as price changes, so that, if the price of the product increases (all other remaining the same), then the supply of the output increases; and if the price of an input increases, then the demand for it decreases. Condition (22) establishes that if the price of an input increases, then the supply of the product decreases. Besides, it shows that the substitution effects are symmetric.

3. Competition versus Profit Maximization

From the discussion above, it is not difficult to see that, if the vector of endowments **e** is large enough, there is a range of **p** such that any price vector in this range is normal to all activity sets $\mathbf{Y}_j$. A price vector that fulfills this condition may be called an equilibrium price vector and it is denoted by **p***.

Now, if $\mathbf{y}_j^* \in \mathbf{Y}_j$ is a maximizer such that $\mathbf{y}_j^* \neq \mathbf{0}$ and **p*** is the equilibrium price vector, the hypothesis of strict convexity ensures that profits in the production of commodity **j** are always positive (see Figure 1). Thus, we have

(22) $\mathbf{p^* y}_j^* > \mathbf{0}$

Figure 1
Equilibrium Prices in a Two-Commodity System
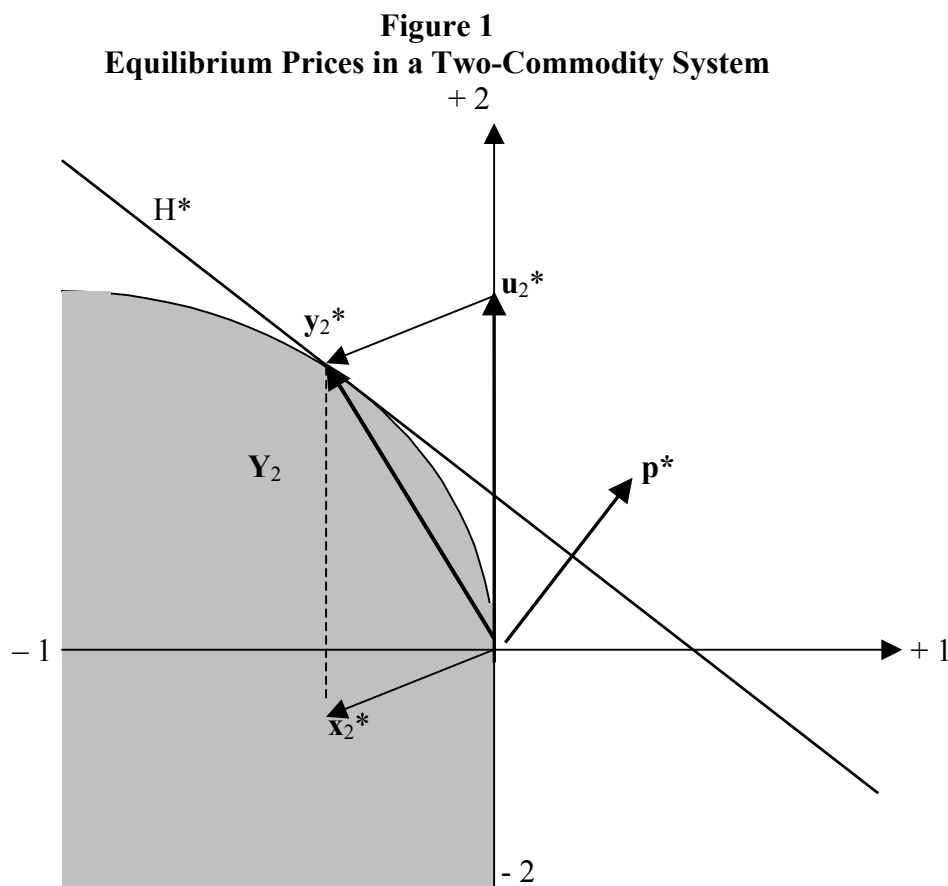
From a neoclassical point of view, the concept of production equilibrium price defined above seems quite unsatisfactory. First, it ignores demand, which is a decisive force determining equilibrium. Besides, condition (22) seems to contradict the

neoclassical tenet that prices are equal to average costs so that economic profits are null in the long run.[5] The first objection, however, can be lifted by including consumers' preference into the analysis so that p^* be defined to fulfill consumers' optimization conditions as well. To address the second objection, we shall stress that capital does not enter the activity set as the other production factors, labor and land. To see why, let's suppose an economy without fixed capital and with only one kind of homogenous labor and one king of homogenous land. Under such assumptions, a point $y_j^* \in Y_j$ is a column of matrix y^*, which may be expressed as:

$$
y^* = \begin{bmatrix}
y_{11}^* & y_{12}^* & \cdots & Y_{1n}^* & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
y_{n1}^* & y_{n2}^* & \cdots & y_{nn}^* & 0 & 0 & 0 \\
y_{L1}^* & y_{L2}^* & \cdots & y_{Ln}^* & y_L^* & 0 & 0 \\
y_{K1}^* & y_{K2}^* & \cdots & y_{Kn}^* & 0 & y_K^* & 0 \\
y_{H1}^* & y_{H2}^* & \cdots & y_{Hn}^* & 0 & 0 & y_H^*
\end{bmatrix}
$$

where y_{Lj}^*, y_{Hj}^*, and y_{Kj}^* are, respectively, the amounts of labor, land, and capital allocated in the production of y_j^*, and y_L^*, y_H^*, and y_K^* are the quantities of these factors supplied in the market.[6]

However, as it has been already pointed out by Walras (1900, p. 212; 215; and 217), capital consists of produced capital goods so that its amount is the inner product of the price vector by the vector of capital inputs, x_j^* (which is non-positive because of our convention that commodity inputs are negative numbers):

(23) $y_{Kj}^* = p^* \, x_j^*$

As a result, it is impossible to solve the maximization problem since the very shape of the production set varies with the price vector.[7]

A way out of the difficulty would be to admit, first, that indeed capital does not enter neither the production set of producer f, Y_{fj}, nor consequently the activity set of commodity j, Y_j, and, second, that the profits identified by (22) would be thus the costs of capital. These assumptions allow us to set the neoclassical condition of long-run equilibrium, according to which prices are equal to costs in all sectors:

(24) $p^* \, [y^* + r \, x^*] = 0$

where y^* is the matrix of net production; x^* is the matrix of commodity and factor inputs used up in the production of y^*, and where r is the price of capital, i.e., the normal rate of rewards of "capital services". Notice that y^* must be redefined so as to write off the line with capital inputs.

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[5] "It is a truth long acknowledged by economists (…) that under certain normal and ideal conditions, the selling prices of commodities are equal to their costs of production" (Walras, 1926, p. 211).

[6] Note that y_{Lj}^*, y_{Hj}^*, and y_{Kj}^* are negative numbers, while y_L^*, y_H^*, and y_K^* are positive numbers.

[7] This seems to be the essence of Joan Robinson's (1953-54) criticism of the concept of production function.

Condition (24) is important, because it shows that, for a given point $\mathbf{y}^* \neq \mathbf{0}$ in the boundary of the total production set $\mathbf{Y}$ and under quite general conditions, the price vector $\mathbf{p}^*$, which would be determined by substituting the value of r in (24), would lie in the left null space of matrix $[\mathbf{y}^* + r\,\mathbf{x}^*]$. In other words, $\mathbf{p}^*$ would be orthogonal to the hyperplane H spanned by the column vectors $[\mathbf{y}_j^* + r\,\mathbf{x}_j^*]$, which passes through the origin $\mathbf{0}$. However, in general, H is not parallel to the hyperplane H_j^* that is tangent to $\mathbf{Y}_j$ at its boundary point $\mathbf{y}_j^*$ (see Figure 2), which means that, in general, $\mathbf{p}^*$ diverges from the equilibrium prices that emerge from the maximization process defined by (18).

Figure 2
Competitive Prices and Equilibrium in a
Two-Commodity System
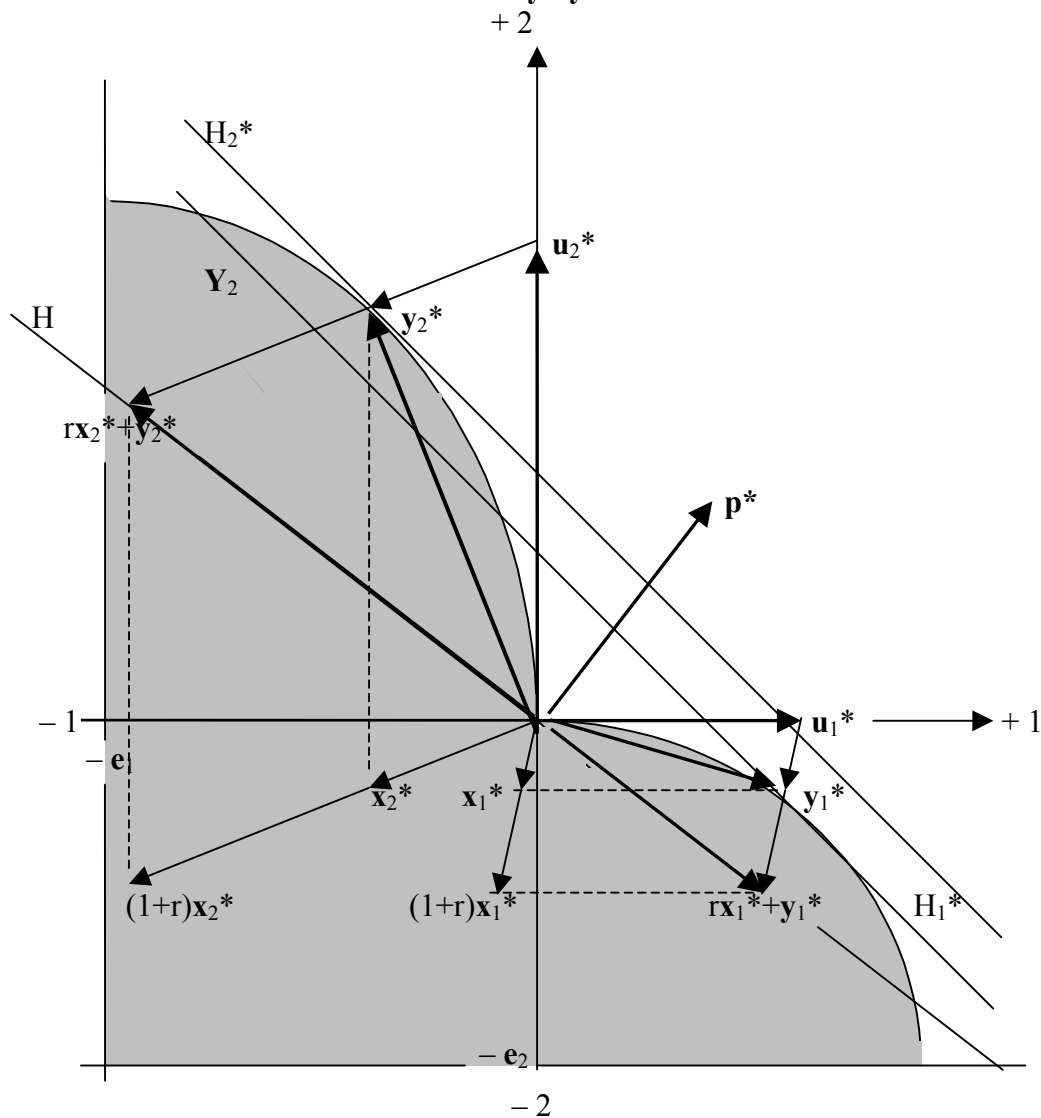
In short, the equilibrium price vector $\mathbf{p}^*$ either satisfies the profit maximization condition (18) such that it is normal to the production set $\mathbf{Y}_j$ for all $\mathbf{j}$ or it satisfies the

competition condition given by (24) such that prices are equivalent to costs and each factor of production has only one price.[8]

4. The Breakdown of the Law of Decreasing Marginal Productivity

As stressed before, one crucial question regarding the neoclassical theory of value and distribution is the law of diminishing returns, which asserts that there is an inverse relationship between the quantity of each factor of production and its marginal productivity.

To simplify the discussion of this question, let us assume that land is a free good and the production set $\mathbf{Y}$ includes only commodities so that the remaining primary factor (labor) is excluded from it. Since there is no fixed capital, equation (24) may be rewritten as:

$$(24a) \quad \begin{bmatrix} \mathbf{p^*} & w & r \end{bmatrix} \begin{bmatrix} \mathbf{y^*} \\ \mathbf{y_L^*} \\ \mathbf{p^*x^*} \end{bmatrix} = \mathbf{0}$$

where $\mathbf{x}$ does not include any primary factor either.

We shall start by asserting that to each point $\mathbf{y^*}$ in the boundary of the production set $\mathbf{Y}$, a unique monotonically decreasing profit-rate curve, r–w can be associated. It can be constructed as follow: by setting $w = 0$ in (24a), the highest rate of return on capital compatible with $\mathbf{y^*}$, which is given by the inverse of the highest characteristic root of matrix $\mathbf{x^* y^{*-1}}$, i.e., by $1/\lambda[\mathbf{x^* y^{*-1}}]$, is obtained. By making $r = 0$, the highest wage rate is obtained as the inverse of the quantity of labor incorporated in the *numeraire* of the price system.[9] In the general case, where $0 < r < 1/\lambda[\mathbf{x^* y^{*-1}}]$, the r-w curve can be derived by resorting to the definition of nominal wage. Considering that $w = \mathbf{p^*d^*}$, where $\mathbf{d}$ stands for the (equilibrium) wage bundle, we obtain the following identity:

$$(25) \quad \mathbf{p^*} [\lambda \mathbf{I} + \mathbf{x^*y^{*-1}} (\mathbf{I} + \mathbf{d^* y_L^*y^{*-1}})^{-1}] = \mathbf{p^*} [\lambda \mathbf{I} + \mathbf{z^*}] = \mathbf{0}$$

where

$$(26) \quad \lambda(\mathbf{z^*}) = 1/r$$

and $\lambda(\mathbf{z^*})$ stands for the spectral radius of matrix $\mathbf{z^*} = \mathbf{x^*y^{*-1}} (\mathbf{I} + \mathbf{d^* y_L^*y^{*-1}})^{-1} =$

$$= \mathbf{x^*y^{*-1}} [\mathbf{I} + (\mathbf{d^* y_L^*y^{*-1}}) + (\mathbf{d^* y_L^*y^{*-1}})^2 + (\mathbf{d^* y_L^* y^{*-1}})^3 + \ldots] \geq \mathbf{0}.$$

If it is assumed that workers prefer consuming more of at least one good or service and no less of any other good or service when they receive a higher real wage, i.e., that $\mathbf{d^*}_2 \geq \mathbf{d^*}_1$ whenever $w_2 > w_1$, it follows that, there is an monotonically inverse relationship between the wage rate and the rate of return on capital. The reason is that

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[8] Hahn (1975, 1982) and Bliss (1975) argue that condition (24) represents a special case of the general equilibrium theory. Nevertheless, it is a necessary corollary of free competition, which establishes that capital leaves the sectors of lower profitability toward sectors of higher profitability. See also Garegnani (1976), Duménil and Levy (1985), Kurz (1985), Kurz and Salvadori (1995).
[9] If $r = 0$, we can conclude from (25) that $\mathbf{p} = - w_{MAX} \mathbf{y_L y^{-1}}$. Since the price of the *numeraire* is unitary by definition and $\mathbf{y_L y^{-1}}$ is the vector of the quantity of labor embodied in the commodities, the conclusion follows naturally.

$\lambda(\mathbf{z^*})$ is an increasing function of the elements of matrix $\mathbf{z^*}$, so that, as a result, $\lambda(\mathbf{z^*}_2) > \lambda(\mathbf{z^*}_1)$ if $\mathbf{d^*}_2 \geq \mathbf{d^*}_1$, i.e., if $w_2 > w_1$. Because of (26), it can be concluded that

(27) $dr/dw < 0$[10]

An alternative way to show the same profit-wage relationship is by expressing the vector of competitive prices as:

(28) $\mathbf{p^*} = -(1 + r)\, w\, \mathbf{y_L^*}\, \mathbf{y^{*-1}}\, [\mathbf{I} + r\, \mathbf{x^*}\, \mathbf{y^{*-1}}]^{-1} = -(1 + r)\, w\, \mathbf{y_L^*}\, \mathbf{y^{*-1}}\, \mathbf{s^*}(r)$

Taking into account that the non negative matrix $\mathbf{s^*}(r)$ can be written as

$\mathbf{s^*}(r) = [\mathbf{I} + r\, \mathbf{x^*}\, \mathbf{y^{*-1}} + r^2\, (\mathbf{x^*}\, \mathbf{y^{*-1}})^2 + r^3\, (\mathbf{x^*}\, \mathbf{y^{*-1}})^3 + \ldots] \geq \mathbf{0}$

it can be seen that $\mathbf{s^*}(r_2) \geq \mathbf{s^*}(r_1)$ if $r_2 > r_1$. Considering, then, that the price of the *numeraire* is unitary ($p_n = 1$), it is not difficult to conclude that the profit rate is a monotonically decreasing function of the wage rate.

From a neoclassical point of view, one problem with wage-profit curves, constructed according to these lines, is that it is assumed that $\mathbf{y^*}$ does not vary with changes in the real wage, which means that each increase in workers' consumption is exactly compensated by the decrease either in entrepreneurs' consumption or in investment such that the net product remains the same. Fortunately, this problem can be addressed through the wage-profit frontier, which is the envelope of all the wage-profit curves that are associated to all boundary points $\mathbf{y^*} \in \mathbf{Y}$: if the net product $\mathbf{y^*}$ varies as a result of a change of the real wage, the resulting new wage rate will belong to another wage-profit curve. This envelope is composed by the outermost points of all the wage-profit curves that represent the highest profit rate that can be achieved by a choice of a feasible production plan, considering a given real wage. Pasinetti (1977, p. 158) calls it the "*technological frontier of possible distributions of income between profits and wages*, or the technological frontier of income distribution possibilities or, even simpler, the technological frontier".

Because each wage-profit curve is monotonically decreasing, so is the wage-profit frontier. But since its slope is the labor/capital ratio, it follows that both reswitching of techniques and inverse capital deepening can be avoided only if the wage-profit frontier is convex, which requires linear wage-profit curves. Considering that

$\mathbf{p^*y^*} + w\, \mathbf{y_L^*} + r\, \mathbf{p^*}(\mathbf{x^*} + \mathbf{d^*y_L^*}) = \mathbf{0}$

the necessary and sufficient condition for having a well-behaved technological frontier is given by:

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[10] Another way is to assumed that the composition of the wage basket $\mathbf{d^*}$ is fixed and rewrite condition (24a) as:

$\mathbf{p}\, [\mathbf{y^*} + \omega\, \mathbf{d^*}\, \mathbf{y_L^*} + r\, \mathbf{x^*}] = \mathbf{0}$

which requires that:

$\det\, [\mathbf{y^*} + \omega\, \mathbf{d^*}\, \mathbf{y_L^*} + r\, \mathbf{x^*}] = \det\, [\lambda\, \mathbf{I} + \mathbf{x^*}\, \mathbf{y^{*-1}}\, (\mathbf{I} + \omega\, \mathbf{d^*}\, \mathbf{y_L^*}\, \mathbf{y^{*-1}})^{-1}] = 0$

where $\mathbf{y_L^*}$ is the row-vector of quantities of labor used up in the production of $\mathbf{y^*}$ and ω is the level of the real wage. By marking the level of the real wage to vary from zero ($\omega = 0$) to its maximum (which is equal to the inverse of the spectral radius of matrix $\omega\mathbf{d^*}\mathbf{y_L^*}\mathbf{y^{*-1}}$), we obtain the wage-profit curve associated with each $\mathbf{y^*} \in \mathbf{Y^*}$.

(29) $\mathbf{y}^* = (1 + R)(\mathbf{x}^* + \mathbf{d}^*\mathbf{y}_L^*)$

where R is a constant, called standard ratio by Sraffa. In other words, the neoclassical law of diminishing returns prevails only if the composition of the net product is the same of the Sraffian standard commodity, whose definition is similar to condition (29). In all other cases, the wage-profit frontier can have either concave or convex segments, like in Figure 3, which means that reswitching of techniques and inverse capital deepening are possible and, as a result, there is no such a thing as the law of decreasing marginal productivity of capital (or labor).

Figure 3
Wage Profit Frontier
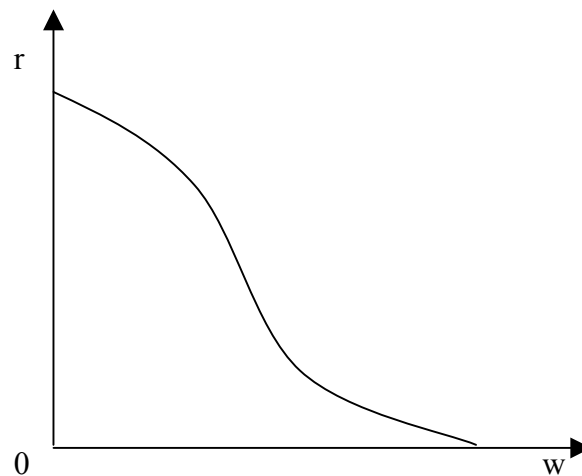
5. **Conclusions**

From the discussion above, it can be concluded that, if the production sets of all commodities are given, the neoclassical concept of free competition is sufficient to have all long-run equilibrium prices determined regardless of profit maximization, which is not necessary for such purposes. Besides, if competition is assumed, the law of diminishing marginal returns can not be derived even if the standard assumptions of the Arrow-Debreu model are accepted. Reswitching of techniques and inverse capital deepening are indeed phenomena that can not be excluded from any meaningful economic model.

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