Divisia Index, Inflation and Welfare[*][†]

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Abstract

This paper addresses a usual criticism in the literature of the welfare costs of inflation, related to the fact that some items in the relevant definition of money pay interests, while others do not. We show that the problem can be solved by using a Divisia index of monetary services as the welfare measure.

1 Introduction

This work aims at investigating, in the search for an adequate measure of the welfare costs of inflation, the fact that some assets included in the theoretically-relevant definition of money pay interest, while others do not. The problem is widespread in the literature. Many analyses define money as a non-interest-bearing asset held by households but, at the same time, use M_{α} ($\alpha \geq 1$) as the respective empirical counterpart.

Referring to the calculation of the welfare costs of inflation, Marty (1999, p.46) notices that:

> "if M1 is used as a relevant money supply, some correction must be made for the interest paid on portions of M1".

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Lucas (2000) also voices concern regarding this fact. Indeed, the connection between the welfare costs of inflation and the Divisia indices of monetary services has been conjectured by this author (p. 270):

> "I share the widely held opinion that M1 is too narrow an aggregate for this period [the 1990s], and I think that the Divisia approach offers much the best prospects for resolving this difficulty".

However, Lucas (2000) does not develop the link between the theoretical measures of the welfare costs of inflation and the Divisia index of monetary services. By developing such a link here, we establish a framework that validates his conjecture.

Our primary purpose in this paper is not empirical. We are solely interested in providing a theoretical investigation of how one could think about the welfare costs of inflation in a model in which different monetary aggregates are used for transacting purposes.

Our results build directly on Simonsen and Cysne's (2001) work, which, in turn, draws upon Lucas (2000). We extend Simonsen and Cysne's original results by investigating situations in which: i) the opportunity costs of all monetary assets are allowed to vary; ii) interest rates are endogenously determined in a general-equilibrium setting and; iii) financial innovations are taken into consideration. As in these previous works, our economy is a deterministic one. Further extensions, including the analysis of risk, are suggested in the Conclusion.

We present our results in two consecutive steps. In the first (Section 6), we assume that the government (here consolidated with the Central Bank) issues all the types of money, either interest-bearing or non-interest-bearing, setting the respective interest rates. Although the objective here is more of a didactic nature, this part of the analysis can be useful as a proxy for situations faced by high-inflation economies[1], or in the case of a banking system facing legal restrictions.

In a second step (Section 7), we close the model by assuming that interest rates on monetary assets are determined by a competitive banking system. Instead of fixing the interest rates, the government is then supposed to fix the reserve requirements on each asset.

The remainder of this work is organized as follows. Section 2 presents the model. Section 3 is used to define three different versions of the Divisia index of monetary services and prove their path independence. Section 4 demonstrates how well the Divisia indices approximate the welfare cost of inflation.

Section 5 shows that financial innovations have a direct negative impact on the welfare costs of inflation. Section 6 exemplifies the use of the different welfare measures investigated here in applied work and briefly discusses the case when the assets' demand functions are not known by the researcher.

Section 7, as previously mentioned, closes the model by introducing a banking system and allowing interest rates to be competitively determined. In this section we also establish sufficient conditions under which the monetary base emerges as the adequate aggregate to be used in the calculations of the welfare costs of inflation[2].

Finally, Section 8 offers the conclusions of the work.

2 The Model

• Households and Firms

Consider an economy where n ($n \geq 1$) different assets can be useful for transacting purposes. We call such assets monetary assets. Households can also hold bonds issued by the government. Such bonds, which are not helpful for transacting purposes, pay the (endogenously determined) benchmark interest rate γ. Each monetary asset is supposed to have, at the margin, a different degree of moneyness.

We denote the monetary assets by the $n-$dimensional vector $X = (X_1, X_2, \ldots, X_n)$, and their real quantities by the vector $x = (x_1, x_2, \ldots, x_n) = (X_1/P, X_2/P, \ldots, X_n/P)$. $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ stands for the interest rate vector associated (in the obvious way) with x. We think of x_1 as currency, in which case $\gamma_1 = 0$. The vector of opportunity costs is defined by $u = (u_1, u_2, \ldots, u_n) = (\gamma - \gamma_1, \gamma - \gamma_2, \ldots, \gamma - \gamma_n)$.

Assumption 1: $x \in R^n_{++}$, $u \in R^n_{++}$.

The infinitely lived representative household is assumed to maximize:

$$\int_0^\infty e^{-gt}\, U(c)dt$$

where $U(c)$ is a concave function of the consumption at instant t and $g > 0$. The household is endowed with one unit of time that can be used to transact or to produce the consumption good, so that $y + s = 1$, where y stands for the production of the consumption good and s for the fraction of the initial endowment spent as transacting time. Note that the product (GDP) is normalized to one when the shopping time is equal to zero.

In their intertemporal utility maximization, households take as a given the nominal interest rate on bonds, γ, and the opportunity costs of holding monetary assets, $u = (u_1, u_2, \ldots, u_n)$. Letting $P = P(t)$ be the price of the consumption good, households face the budget constraint:

$$\sum_{i=1}^{n} \dot{X}_i + \dot{B} = \gamma B + \langle \gamma, X \rangle + P(y - c) + H$$

where H indicates the (exogenous) flow of income transferred to the household by the government. Making $\pi = \dot{P}/P$ (inflation rate), $\gamma_R = (\gamma_1 - \pi, \gamma_2 - \pi, \ldots, \gamma_n - \pi)$ and $h = H/P$, the budget constraint reads:

$$\dot{b} + \sum_{i=1}^{n} \dot{x}_i = 1 - (c + s) + h + (\gamma - \pi)\, b + \langle \gamma_R, x \rangle \tag{1}$$

The consumer is also subject to the transacting-technology constraint:

$$c = N(x, s) \tag{2}$$

Assumption 2: The transacting technology $N(x, s)$ is blockwise-weakly separable with respect to the vector x and the variable s.[3]

We assume the particular case of separability:

$$c = N(x, s) = G(x)\varphi(s). \tag{3}$$

with $\varphi(0) = 0$, $\varphi'(s) > 0$, $\varphi''(s) \leqq 0$.

Assumption 3: The monetary aggregator function $G(x)$ is differentiable, first-degree homogeneous, and strictly increasing in each of its variables, with decreasing marginal returns.

In the steady state, necessary conditions for optimization are given by the equilibrium equation (4) and by the first order conditions (5) and (6) below:

$$1 - s = G(x)\varphi(s) \tag{4}$$

$$\gamma = \pi + g \tag{5}$$

$$G_{x_i}(x)\varphi(s) = u_i\, G(x)\, \varphi'(s), \quad i = 1, 2, \ldots, n \tag{6}$$

Equation (5) establishes the link between the rate of inflation and the benchmark interest rate. Given $G(x)$ and $\varphi(s)$, equations (4) and (6) can be used to determine the $n + 1$ variables $u(x)$ and $s(x)$, in which case the respective Jacobian is a positive definite diagonal matrix.

Notice, though, that the function $\varphi(s)$ is generally not known by the researcher. Therefore, the above equations do not allow a direct determination of s, the variable in which we are interested. We shall return to this question in Section 5, where an indirect way of finding $s(x)$ is devised.

By using the homogeneity of G and Euler's theorem, one can write:

$$\varphi(s) = \langle u, x \rangle \varphi'(s) \tag{7}$$

a result that we shall use later.

- Government

As in Lucas (2000), we consider this to be an economy with lump sum taxation. In the analysis of Section 6, where the government is supposed to issue all of the monetary assets, and in the steady state:

$$h = -(\gamma - \pi) b - \langle \gamma_R, x \rangle \tag{8}$$

Note that, except in the case of x_1; for which $\gamma_{R1} = -\pi$; the vector of real interest rates γ_R above can assume positive or negative values, depending if the nominal interest rates of the respective monetary asset has been fixed below or above the rate of inflation.

In Section 7 the government is supposed to issue just currency and bonds. Other monetary assets are issued by a competitive banking system, and their respective interest rates are endogenously determined. In this case, (8) reduces to

$$h = -(\gamma - \pi) b - \pi z$$

where z stands for the real value of the monetary base.

3 Divisia Indices and Path Independence

Divisia indices have been proposed by Barnett (1980) as the adequate way to build monetary aggregates.

While conventional (simple-sum) monetary aggregates are not useful for welfare measurements, Divisia aggregates can perform such a function. As argued by Bruce (1977), there is a general equivalence between Divisia quantity indices and consumer's surplus measures of welfare losses. A particular version of this general principle associates Divisia indices of monetary services with the welfare costs of inflation.

Nominal Divisia indices weigh the variations of the quantities of each monetary aggregate by its relative opportunity costs. In equilibrium, these opportunity costs are equivalent to prices, and the result is a multidimensional consumer's surplus measure. In economies where currency and other monies perform monetary services, components with high opportunity cost, which are the ones most frequently used for purposes of transaction (currency being a superior limiting case), are given a higher weight in the Divisia methodology. On the other hand, components with low opportunity costs (those that pay an interest rate close to the benchmark interest rate), which are the ones more likely to be held for saving services, rather than for transactions, are given a reduced weight. In this way, Divisia aggregates adequately capture the transacting motive for holding money, which, in turn, can be associated with welfare measures.

Formally, the Divisia index is a map from the set of paths in R^n into the real line: Different versions of it can be found in the literature, depending upon how the nominal prices used in their construction are normalized or deflated (Bruce (1977)). In this work, we work with three different versions of such indices (two of which can be used as welfare measures), based on different normalization.

We consider continuously differentiable paths $\chi : [0, 1] \to R^n_{++}$ followed by the vector of monetary aggregates x and define:

Definition 1: Divisia S (DS).

Given the map defined on R^n_{++}:

$$F_S(u(x)) = \left(\frac{u_1}{\langle u, x \rangle}, \frac{u_2}{\langle u, x \rangle}, \ldots, \frac{u_n}{\langle u, x \rangle}\right)$$

we define DS as:

$$DS(\chi) = \exp \int_{\chi} \langle F_S(u(x)), dx \rangle \tag{9}$$

This is the original version of the Divisia (1925) index.

Definition 2: Divisia E (DE).

Alternatively, we make

$$F_E(u(x)) = \left(\frac{u_1}{1 + \langle u, x \rangle}, \frac{u_2}{1 + \langle u, x \rangle}, \ldots, \frac{u_n}{1 + \langle u, x \rangle}\right) \tag{10}$$

and define DE by:

$$DE(\chi) = \int_{\chi} \langle F_E(u(x)), dx \rangle \tag{11}$$

This version of the Divisia index is found in Simonsen and Cysne (2001).

Definition 3: Divisia G (DG).

Third, we define F_G by

$$F_G(u(x)) = (u_1; u_2; \ldots; u_n)$$

and make:

$$DG(\chi) = \int_{\chi} \langle F_G(u(x)); dx \rangle \tag{12}$$

This version of the Divisia index is presented, for instance, in Bruce (1977).

• Path Independence

As line integrals, Divisia indices can suffer from the serious defect of depending on the path over which integration is taken. We shall see in this section that, given the assumptions of our model, all three versions of the Divisia index here presented are path independent.

The DS version of the Divisia index exactly tracks the associated aggregator G evaluated at the optimum, which implies its path independence. Indeed, one can easily check that $\log G(x)$ is a potential function for the vector field given by $F_S(u(x))$ in (9)[4].

Proposition 1 - The Divisia indices DG, DS and DE are path independent.

Proof. From (6), $u_i = G_{x_i}(x)\phi(s)/G(x)\phi'(s)$ and $u_j = G_{x_j}(x)\phi(s)/G(x)\phi'(s)$ imply $\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i}$ for all $i; j; i \neq j$. Locally, this is a sufficient condition for path independence of DG. To obtain the same result with respect to DE (or DS) it suffices using (7) to substitute $u_j G(x)/G_{x_j}(x) = u_i G(x)/G_{x_i}(x) = \phi(s)/\phi'(s)$ for $\langle u; x \rangle$ in (9) and (11), and noticing, again, that the cross derivatives of the Divisia weights are equal. ■

4 The Relation Between the Shopping Times and the Divisia Index DE

The reason why inflation leads to welfare losses in our model is the usual one found in shopping-time economies. Households can acquire monetary services by holding different types of monetary assets. When inflation rates increase, the opportunity cost of holding monetary assets also increases (either when interest rates on monetary assets are exogenously or competitively determined), leading households to spend more time shopping, as a counterpart to lower transaction balances. Consequently, the remaining time allocated in the production of the consumption good decreases, and so does welfare.

In the model presented in Section 2, s denotes the percentage reduction in production and consumption when the economy is not completely satiated with monetary services, and represents a direct measure of the welfare costs of interest rate wedges, as a fraction of the product. For empirical purposes, however, the function $\phi(s)$ is not known, and one cannot directly calculate s.

Following Lucas (2000) and Simonsen and Cysne (2001), we recover s abstracting from the knowledge of the demand functions only. In a subsequent step, we show that the Divisia index DE is a good approximation for s.

Start by totally differentiating (4) and using the n first order conditions (6) to get:

$$-ds = G(x)\,\phi'(s)\,(\langle u, dx\rangle + ds)$$

Use (7) and (4) to eliminate $\phi(s)$ and $\phi'(s)$;

$$ds = -\frac{(1-s)\langle u, dx\rangle}{1-s+\langle u, x\rangle} \tag{13}$$

Equation (13), presented in Simonsen and Cysne (2001), is an n-dimensional version, for an economy with n types of monies, of the expression that Lucas (2000. eq. 5.8) derives in his work, in the particular case when $n = 1$ (with $x_1 = M_1/P$; M_1 standing for the usual definition of the means of payment).

When $n > 1$; (13) represents a system of n simultaneous non-separable and non-linear partial differential equations:

$$\frac{\partial s(x)}{\partial x_i} = V_i(s(x); x) \qquad i = 1, 2, ..., n$$

where

$$V_i(s(x); x) = -\frac{(1-s(x))\,u_i(x)}{1-s(x)+\langle u(x), x\rangle} \tag{14}$$

A solution to this system is a function $s(x)$ that satisfies these equations identically in x. For $n > 1$; one should be aware that, if the functions $u_i(x)$ are arbitrarily-assigned, this total differential equation does not necessarily correspond to a primitive $\xi(s; x) = c$. This imposes integrability conditions on the demand functions that can be empirically tested in order to verify the reasonableness of our model.

Such integrability conditions originate from the symmetry of the cross partial-derivatives of $s(x)$ for $i = 1, 2, ..., n$; $j = 1, 2, ..., n$; $i \neq j$:

$$s_{x_i x_j} = \frac{\partial V_i}{\partial s}\frac{\partial s}{\partial x_j} + \frac{\partial V_i}{\partial x_j} = s_{x_j x_i} = \frac{\partial V_j}{\partial s}\frac{\partial s}{\partial x_i} + \frac{\partial V_j}{\partial x_i} \tag{15}$$

As one observes from (15), when V can be made not to depend upon s, as in

$$\tilde{V}_i = \tilde{V}_i(x)$$

the integrability conditions turn out to the simpler form:

$$s_{x_i x_j} = \frac{\partial \tilde{V}_i}{\partial x_j} = s_{x_j x_i} = \frac{\partial \tilde{V}_j}{\partial x_i} \tag{16}$$

In our case, it follows from (14) that V does depend on s, therefore characterizing a non-separable equation. However, we shall see below that the additive symmetric of DE $(-DE)$, obtained from (11), can be used as a reasonable approximation to s. Using DE, instead of s, presents the following nice features: i) once the demands for the monetary assets are known, assuming they satisfy (16), the attainment of closed-form solutions for the welfare measures is algebraically simpler (as long as solving a line integral is easier than solving a system of non-separable partial differential equations); and ii) alternatively, when the demands for monetary assets are not known by the researcher, using DE, which is an index, has the advantage of allowing for direct welfare calculations from market data. In particular, as demonstrated in Section 6, this allows a welfare-ranking of interest rate vectors.

Our demonstration that DE is a good approximation for s differs from the equivalent one in Simonsen and Cysne (2001) by explicitly considering n assets (instead of just two) and, more important, by allowing the opportunity cost of more than one asset to change.

Proposition 2 - (Generalization of Simonsen and Cysne (2001)): Consider paths $\psi^* : t \to x(t)$, $t \in [0,1]$, $\psi^*([0,1]) \subset R^n_{++}$, with $\langle u, dx\rangle < 0$ (a particular case occurs for paths ψ^{**} with $\lim_{t \to 0} x_i = +\infty$ and $x_i'(t) < 0$, $i = 1, 2, ..., n$). Let $s(x)$ denote the solution to (13) along such a path. Then,

(1)

$$1 - e^{DE(x)} < s(x) < -DE(x) \tag{17}$$

(2) For values of DE sufficiently low,

$$\frac{-DE(x) - (1 - \exp^{DE(x)})}{s(x)} \approx \frac{-DE(x)^2}{2s(x)} \approx \frac{-DE(x)}{2} \tag{18}$$

Proof. See Appendix. ■

Remark 1 - Since DE is path independent, the condition $\langle u, dx\rangle < 0$ can be dealt with by considering adequate finite sequences of paths in R^n_{++}.

Remark 2 - Since $\langle u, x\rangle > 0$, inequality (17) can be immediately extended to $1 - e^{DE(x)} < s(x) < -DE(x) < -DG$.

5 Financial Innovations

The above development has been made under the assumption that the monetary aggregator is unchanging. Here we consider the case of a non-neutral technological progress of the financial technology and analyze how the measures of the welfare costs of inflation should be adjusted. A non-neutral progress, for instance, allows us to encompass possible M_1-saving innovations in the transacting technology, as seems to have happened in the 80s and 90s. The analysis can be easily accomplished by assuming a transacting technology given by

$$G(\tilde{x}) = G(\delta_1 x_1, \delta_2 x_2, \ldots, \delta_n x_n)$$

where $\tilde{x}_i = \delta_i x_i$, $i = 1, 2, \ldots, n$, with the variables δ_i allowed to vary in order to translate productivity variations. Proceeding the maximization of utility as before, the first order equations remain the same. Besides, since $G_{x_i} x_i = G_{\tilde{x}_i} \tilde{x}_i$ and $G_{\tilde{x}_i} d\tilde{x}_i = G_{x_i} x_i (dx_i/x_i + d\delta_i/\delta_i)$, one easily obtains:

$$ds = -\frac{(1-s)\sum_{i=1}^{n} u_i x_i (dx_i/x_i + d\delta_i/\delta_i)}{1 - s + \langle u, x \rangle}$$

of which, for paths satisfying $\sum_{i=1}^{n} u_i x_i (dx_i/x_i + d\delta_i/\delta_i) < 0$,

$$dDE = -\frac{\sum_{i=1}^{n} u_i x_i (dx_i/x_i + d\delta_i/\delta_i)}{1 + \langle u, x \rangle} \qquad (19)$$

is a good approximation.

Proposition 3: In the case of non-neutral financial innovations, the same Divisia weights $1/(1 + \langle u, x \rangle)$ defined in (10) can be used in the calculations of the welfare costs of interest-rate wedges or inflation. However, a second term, $(x_i d\delta_i/\delta_i)$, which depends on the rate of growth of the productivity of each asset (as well as of the level of each asset), must be added in the weighed sum[5].

Financial innovations, as displayed by (19), have a direct negative effect on the welfare costs of inflation. This point explains why countries under severe inflationary processes usually present a high demand for financial innovations, mainly those which allow households to economize on currency and noninterest-bearing demand deposits (where the weights are higher).

6 Measuring Welfare Costs in Empirical Research

As remarked in the Introduction, our purpose in this section and the next is not quantifying the welfare costs of inflation for particular economies. The simulations only aim to clarify how our theoretical results could be used in practice. We also remind the reader that we have already derived, analytically, a maximum relative error of $-DE$; as an approximation to s (see (18)), which replaces the usual sensitivity analysis.

• Ranking Interest-Rate Wedges

When there is only one interest rate, and therefore only one opportunity cost to be considered, the Friedman rule, when valid[6], states that the social optimum is achieved by making the interest rate (and the opportunity cost of currency) equal to zero. The same type of rule applies, multidimensionally, in our model. If all opportunity costs tend toward zero, s tends towards zero.

However, in economies where several opportunity costs are considered, it is not always clear which situation leads to a higher or lower welfare, since some costs can increase, while others decrease. In this case, the Divisia indices presented here can be used as a device for reducing the comparison between two different opportunity costs vector (or two different monetary assets vector) to a single scalar.

In the example below we analyze a case in which the vector of opportunity costs has changed from $u(x(0))$ to $u(x(1))$: We assess the welfare variation using (13), (11) and (12), as well as the lower bound for s $(1 - e^{DE(x)})$ presented in (17) .

Example 1 - We consider an economy where $\phi(s) = s$ and the transacting technology (which is not known by the researcher) is given by:

$$\begin{aligned} c &= G(x)s = A x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} s \\ 1 &= a_1 + a_2 + \dots + a_n; \; A > 0 \end{aligned}$$

We assume, in addition, that the researcher has been able to properly estimate the demand functions compatible with this technology (the case when the demands are not known is explored later):

$$u_i = \frac{a_i / x_i}{1 + G}; \quad i = 1, 2, \dots, n \tag{20}$$

Our objective is recovering the measure s (or some good approximation of it) abstracting from the knowledge of these demands, but not of the function

$\psi(s)$. The first option is plugging equations (20) directly into (13) and solving the system of non-separable partial differential equations given by:

$$ds = -\frac{(1-s)\sum_{i=1}^{n}\frac{a_i dx_i}{x_i(1+G)}}{1-s+\langle u, x\rangle}$$

which in this case leads to the closed-form solution:

$$s(x) = \frac{1}{1+G(x)} \tag{21}$$

One possible problem with this alternative is that providing a closed-form solution to the non-separable partial differential equation (13) can be a non-trivial task, depending on the assets demand functions that are plugged into (13)[7]. Alternatively, one can use the approximation DE, given by (11), a procedure which is allowed by (18). In this specific example, using (20) in (11):

$$DE(x) = -\frac{1}{2}\log(1+\frac{2}{G}) \tag{22}$$

For the purpose of comparison, we also present the expression for DG:

$$DG(x) = -\log(1+\frac{1}{G}) \tag{23}$$

In order to illustrate these results with a numerical example, consider a situation where $n = 3$; $A = 2000$; $a_1 = 0.5$; $a_2 = 0.3$; $a_3 = 0.2$. We assume the initial values of the monetary aggregates to be given by $x_1(1) = 0.090$; $x_2(1) = 0.058$; $x_3(1) = 0.045$; and the final values by $x_1(2) = 0.053$; $x_2(2) = 0.032$; and $x_3(2) = 0.022$. The implied values of the opportunity costs in the first and second situations are given by, respectively, $u(1) = (4.0156\%; 3.7387\%; 3.2125\%)$ and $u(2) = (12.1857\%; 12.1095\%, 11.7426\%)$. In this case we get, as a percentage of the product:

	$1-e^{DE}$	s	$-DE$	$-DG$
Initial	0.7202	0.7228	0.7228	0.7254
Final	1.2834	1.2917	1.2918	1.3001
Variation	0.5632	0.5689	0.5689	0.5747

If the researcher chooses to solve the non–separable partial differential equation, he will find a welfare cost figure, due to the change of the vector of opportunity costs, from $u(1)$ to $u(2)$, of 0.56886% of GDP. Alternatively, the use of the Divisia index $-DE$, leads to the figure of 0.56892% of GDP, a negligible difference. In any case, as claimed, the Divisia methodology allows a ranking of the different interest vectors.

The Case of Unknown Demands

In the example above, we assumed the exact functional specification and the parameters of the assets demand functions to be known by the researcher. When this is possible, one only needs to rely on quantity data and on the estimated parameters of the underlying demand functions.

When this is not possible, or is too costly, the results derived here can be useful by considering the discrete version of (11). Indeed, one nice feature of using the Divisia index DE as a welfare measure is that it can always be computed, given observations on interest rates and monetary aggregates. Statistical index numbers do not depend on any unknown parameters. The use of market prices compensate for the absence of knowledge about parameters or functional specifications. Prices (here, opportunity costs) and quantities have the advantage of being directly observable.

Since collecting data in continuous time is impossible, one has to rely on some approximation of (11) defined in discrete time. DED, below, provides one such possible approximation:

$$DED(t) - DED(t-1) = \sum_{j=1} w^*_{t,j}\left[x_{j,t} - x_{j,t-1}\right] \tag{24}$$

where

$$w^*_t = \frac{1}{2}\left(\frac{u_1}{1+\langle u, x\rangle}(t) + \frac{u_1}{1+\langle u, x\rangle}(t-1)\right)$$

DED consistently approaches DE as Δt goes to zero. If we use this formula to make a rough approximation of DE, based only on the initial and final values of the variables, we get, using the parameter values of our previous example, $\Delta DED = -0.6728$, as against the value $\Delta DE = -0.5689$ previously calculated. The approximation can always be improved by the use of additional quantity and price data observations between the two periods of reference.

Measuring the Welfare Costs of Inflation

It follows from (5) that our economy is a Fisherian one, where the benchmark interest rate is determined by the rate of inflation, which is endogenous in the model, and by the rate of time preference. In this case, since we are assuming so far that the interest rates of the monetary assets are exogenously determined by the government, the interest-rate wedges are directly linked to the inflation rate (Section 7 deals with the case when the interest-rate wedges are endogenously determined in a competitive economy).

Example 2 - We use the same particular transacting technology, the same demand functions and the same values of the parameters n; A; a_1; a_2; and a_3 of example 1. We initiate the table below assuming the economy to be satiated with monetary services for an annual rate of deflation equal to 2%: We then make the annual inflation rate (π) vary from 0.0% to 2.0 (200%). With the rate of time preference $\rho = 0.02$, the nominal interest rate of the benchmark asset varies from 0.02 to 2.02. Since currency (x_1 in this example), by definition, pays a nominal interest rate equal to zero, its opportunity cost (u_1) will also vary in the same range. The two other assets, x_2 and x_3; are assumed to pay annual fixed nominal interest rates equal to, respectively, 0.003 (0.3%) and 0.008; (0.8%); in which case their opportunity costs will vary from 0.017 to 2.017 (x_2); and from 0.012 to 2.012 (x_3); respectively. The table below presents the values of the different measures of the welfare costs of inflation as a percentage of GDP.

Inflation	$1 - e^{DE}$	s	$-DE$	$-DG$
-2%	0.0000	0.0000	0.0000	0.0000
0%	0.4883	0.4895	0.4895	0.4907
5%	0.9623	0.9669	0.9669	0.9716
10%	1.2662	1.2742	1.2743	1.2824
20%	1.7151	1.7298	1.7300	1.7449
50%	2.6213	2.6557	2.6563	2.6916
100%	3.6376	3.7038	3.7054	3.7741
150%	4.4073	4.5043	4.5073	4.6089
200%	5.0481	5.1753	5.1800	5.3141

One can observe that the difference between s and $-DE$ is immaterial, even for values of $-DE$ not so close to zero.

Besides the convergence of $1 - e^{DE}$; s; $-DE$ and $-DG$ to zero, for low rates of inflation, it becomes clear that, as inflation rises, both the difference between s and $1 - \exp(DE)$; and between $-DG$ and s, increase. The same happens to $-DE - s$, but at a significantly lower rate.

7 Closing the Model

In this section we drop the assumption that all near monies are issued by the government. Instead, we close the model by hypothesizing that the monetary assets other than currency are issued by a competitive banking system. Banks are supposed to buy bonds from the government and to sell monetary assets to the households. Government issues currency and bonds and collects a (fixed) fraction of reserve requirements on deposits, over which it pays zero nominal interests. k_{x_i} stands for the reserve requirements on deposits x_i. Banks buy bonds that pay an interest rate γ and, operating competitively, pay an interest rate on deposits given by $\gamma_{x_i} = (1 - k_{x_i})\gamma$. The interest rate wedge of deposit x_i is then equal to $u_i = \gamma - \gamma_{x_i} = k_{x_i}\gamma$.

Bailey (1956, p. 104), analyzed the effect of banks on the social costs of inflation under two distinct polar situations. In the first case, described by him as that when "banks operate rationally", the only money used for means of payment is interest-bearing bank deposits. In such a case, only the monetary base (then equal to the reserve requirements times the volume of outstanding deposits) would matter for the calculation of the welfare costs of inflation. Bailey also analyzes a second case, in which banks would not charge the economic rate of interest for their loans, and would not pay market interest for their deposits (which would correspond to what he describes as the non-rational situation). In this second case, he argues that the welfare cost of inflation is the same for a given rate of inflation "regardless of what fraction of the money supply is currency".

Parallel to Bailey's analysis, Proposition 4 below establishes sufficient conditions under which the knowledge of the demand function for the monetary base alone provides all the necessary data for the calculation of the welfare costs of inflation.

Given the vector of reserve-requirement ratios $\bar{k}$, the interest rate-wedge vector (or price vector) $(u(t = 1) = \bar{k}\gamma_1)$ is always proportional to some fixed base price vector $(u(t = 0) = \bar{k}\gamma_0)$, leading, therefore, to a particular case of a composite commodity (Hicksian separability).

We make x_1 in (11) stand for currency (m), with an interest rate $\gamma_1 = 0$, and an opportunity cost $\gamma - \gamma_1 = \gamma$. For all deposits, the multiplication of the interest rate wedge u_j $(j = 2, ..., n)$ by the infinitesimal variation of the respective monetary aggregate (the term $u_j dx_j = k_j \gamma dx_j$ in (11)), exactly equals the benchmark interest rate (γ) times the infinitesimal variation of the fraction of the asset that is maintained as a monetary liability of the Central Bank, as reserve requirements $(u_j dx_j = (k_j\gamma)dx_j = \gamma d(k_j x))$. Therefore, the welfare costs of interest rate wedges can be properly measured considering only the monetary base (z) and the benchmark interest rate γ.

Proposition 4 - Consider an economy where, besides currency, issued by the Central Bank, there are $n - 1$ $(n \geq 2)$ monetary assets that can be used for transacting purposes, each one paying a different interest rate. Also, assume that the interest rates paid by theses monetary assets are competitively determined by a banking system that takes the interest rate on bonds and the (fixed) reserve-requirements vector as given. Then, the welfare costs of inflation can be adequately measured by the expression

$$-DE(u(x)) = -\int_{\chi} \frac{\gamma}{1+\gamma z} dz \tag{25}$$

where z stands for the real value of the monetary base and γ for the nominal interest rate.

Proof. First, notice that the condition $\langle u, dx \rangle < 0$ of Proposition 2 is trivially satisfied in this case, provided that the demand for the monetary base is a decreasing function of the interest rate. Using DE as an approximation for s,

$$\begin{aligned} -DE(u(x)) &= -\int_{\chi} \sum_{j=2} \frac{\gamma dm + k_j \gamma dx_j}{1+\gamma m + \sum_{j=2}^{n} \gamma k_j x_j} \\ &= -\int_{\chi} \sum_{j=2} \frac{\gamma (dm + k_j dx_j)}{1+\gamma (m + \sum_{j=2}^{n} k_j x_j)} \\ &= -\int_{\chi} \frac{\gamma}{1+\gamma z} dz \end{aligned}$$

■

Remark 3 - Note two differences between such results and Bailey's analysis of the "rational" case. First, the general-equilibrium approach to the problem leads to an endogenous determination of an integrating factor ($\frac{1}{1+\gamma z}$) which is not present in the partial-equilibrium analysis. Second, in our framework the monetary base is composed not only of reserves, as in the polar case analyzed by Bailey, but also of currency.

Remark 4 - The conditions of Proposition 4 are somewhat restrictive, indicating that the reduction of the general expression of DE given by (11) to (25) is not necessarily valid as a general statement.

When the conditions established in Proposition 4 are valid, welfare measures based on M_1 can overestimate the welfare cost of inflation by a factor close to the value of the monetary multiplier (around three in the United States)[8]. The basic reason is that by using the base, instead of M_1, one implicitly recognizes the fact that banks return part of their income to the holders of monetary assets, reducing distortions caused by inflation. This result has been illustrated by Bali (2000) and by Marquis (1999), and is likely to be robust.

8 Conclusions and Further Directions

We have considered economies where interest-bearing and non-interest bearing monetary assets are used for transacting purposes and have concluded that a specific Divisia index emerges as the correct welfare measure for the welfare costs of inflation. We have also shown that financial innovations have a direct negative impact on the welfare costs of inflation, and derived an expression that shows how to take non-neutral financial innovations into consideration in welfare measurements. Finally, we derived sufficient conditions under which the calculation of the welfare costs of inflation using the Divisia methodology demands only the knowledge of the demand for the monetary base.

A nice feature of using an index number is that it allows for welfare comparisons even when the demand functions for the monetary assets are not known by the researcher. When the demand functions are known, the path independence of the respective line integral ensures that only the initial and final values of the monetary aggregates vector will suffice for welfare measurements of interest-rate wedges or inflation.

One limitation of the investigations developed here is not allowing for uncertainty. There is now substantial literature on extending Divisia monetary aggregation to the case of risk. Barnett and Serletis (2000, Section 3) includes reprints of four important papers in the area. Barnett (1995) shows that the exact tracking property of the Divisia index continues to hold in this case, provided that the only risk is relative to future prices and interest rates. This conclusion might suggest that the investigations here developed are not affected by this literature (at least in the continuous-time approach), but a direct analysis of the model under risk remains to be explored. Another recent paper in the area is Barnett, Hinich and Piyu Yue (2000).

A second possible extension relates our investigations with the literature on the non-payment of interest on required reserves and the size of tax implied on banks. Barnett, Hinich and Weber (1986) have investigated this point.

Appendix

Proof. (Proposition 2)

(a) Along the paths χ^* considered, since $0 < s < 1$ and $\dot{x}(t) < \bar{0}$, we can write:

$$
\begin{aligned}
\int_{\chi^*} ds &= \int_a^b -\frac{\langle u(x(t)), \dot{x}(t)\rangle}{1 + \langle u(x(t)), x(t)\rangle/(1 - s(x(t)))}dt \\
&< \int_a^b -\frac{\langle u(x(t)), \dot{x}(t)\rangle}{1 + \langle u(x(t)), x(t)\rangle}dt \\
&< \int_a^b -\frac{\langle u(x(t)), \dot{x}(t)\rangle}{1 - s(x(t)) + \langle u(x(t)), x(t)\rangle}dt \\
&= \int_{\chi^*} \frac{ds}{1 - s}
\end{aligned}
$$

(17) follows from the above inequalities by noticing that: (1) the third term in the above expressions is equal to $-DE$; (2) $\lim_{x\to\infty} s = 0$ and (3) $\int_{\chi^*} \frac{ds}{1-s} = -\ln(1-s)\,\big|_{\infty}^{x} = -\ln(1-s(x))$.

(b) The second part of Proposition 2 (equation (18)) is obtained by taking a second-order Taylor approximation to the exponential function. This makes $-DE - (1 - \exp(DE)) \approx DE^2/2s$. Using L'Hôpital's Rule in (11) and (13), one concludes that DE/s tends toward one when x (or any of its components) tends toward infinity. Therefore, as the components of x increase, $DE \searrow 0$ and $DE^2/2s$ tends towards $DE/2$. ■

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Notes

[1]As Calvo and Végh (1996, pp. 1) observe: “In high in‡ation countries, policymakers often end up paying interests on part of the money supply”.

[2]The literature takes di¤erent approaches with respect to this issue (e.g. Aiyagari, Braun and Eckstein (1998) use the monetary base, whereas Lucas (2000) uses M_1 and Cooley and Hansen (1991) use a measure of “the portion of M_1 that is held by households”).

[3]For an analysis of how multidimensional partial-equilibrium measures of the welfare costs of in‡ation can fail to be integrable without this hypothesis, see Cysne (2001).

[4]Divisia (1925, pp. 43, footnote 1), discusses the equivalence between the Divisia index and a curve integral. The demonstration of the path independence of DS can be found in Hulten (1973), for the case of a general, weakly-separable, linear-homogenous function. However, it was only after Barnett’s (1980) derivation of the user-cost price of monetary services that the result could be extended to monetary aggregation.

[5]Caves, Christensen and Diewert (1982) provides extensions of index numbers to technological change. Spencer (1998), using a static cost-minimization argument, arrives at an expression similar to (19) regarding the Divisia index DS.

[6]See Woodford (1990) for a description of some models that lead to this result.

[7]We have been able to do it without further problems because we actually made the inverse way of departing from the knowledge of the transacting technology, a procedure that is not allowed to the researcher.

[8]Suppose that the money demand for M_1=P (m) and for the monetary base are given, respectively, by:

$$\begin{aligned} m &= A°^{¡a} \\ z &= A^{¤}°^{¡a^{¤}} \end{aligned}$$

In this case the (11) leads, respectively, to:

$$\begin{aligned} ¡DE(r) &= \frac{a}{1 ¡ a}\ln(1 + A°^{¡a}) \\ ¡DE^{¤}(r) &= \frac{a^{¤}}{1 ¡ a^{¤}}\ln(1 + A^{¤}°^{¡a^{¤}}) \end{aligned}$$

depending, respectively, if M_1 or the monetary base are used in the welfare calculations (DE and $DE^{¤}$ are de…ned in the obvious way). For a ¼ $a^{¤}$ and low interest rates, $DE=DE^{¤}$; has a value close to the value of the money multiplier.